Hydrostatic shape of the Earth up to the second order

Frédéric Chambat, Ecole Normale Supérieure de Lyon,
December 2011

Tutorial for the MATLAB programm `hydrostatic.m` for numerical calculations of the Earth's Flattening up to second order and to be downloaded at :
`http://frederic.chambat.free.fr/hydrostatic/`
Main reference :
Chambat F., Ricard Y., Valette B., 2010, Flattening of the Earth: further from hydrostaticity than previously estimated, Geophys. J.Int., 183, 727-732.

Contents

1 Notations

In the hydrostatic hypothesis the surfaces of equal density are also surfaces of equal pressure and equal gravity potential. Let $s = s(r, \theta)$ be the shape of such a surface, where : s is the distance from the Earth center, r the mean radius of s, θ the colatitude (s does not depend on the longitude). Up to the second order one can put s in the form:

$$s(r, \theta) = r\,\{\,1 + f_2(r)P_2(\cos\theta) + f_4(r)P_4(\cos\theta)\,\}\,, \tag{1}$$

where the P_n are Legendre polynomia of degree n, e.g.:

$$P_0(\cos\theta) = 1, \quad P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1), \tag{2}$$

$$P_4(\cos\theta) = \frac{1}{8}(35\cos^4\theta - 30\cos^2\theta + 3). \tag{3}$$

In order to compute the $f_n(r)$ one needs to know:

- Ω the angular speed of the Earth,

- GM the geocentric gravitational constant,
- $\rho(r)$ the density distribution of an Earth reference (spherical) model, given up to the external radius R.

Define the mean density within sphere or radius r:

$$\bar{\rho}(r) = \frac{3}{r^3} \int_0^r \rho(y) y^2 \, \mathrm{d}y, \tag{4}$$

The solution depends on the two following functions only. A 'density factor':

$$\gamma(r) = \frac{\rho(r)}{\bar{\rho}(r)}, \tag{5}$$

and a 'rotationnal factor':

$$\tilde{m}(r) = \frac{\Omega^2 R^3}{GM} \frac{\bar{\rho}(R)}{\bar{\rho}(r)} = m \frac{\bar{\rho}(R)}{\bar{\rho}(r)} \tag{6}$$

with G the gravitational constant, $m = \tilde{m}(R)$ defined by:

$$m = \frac{\Omega^2 R^3}{GM}. \tag{7}$$

On can also write the last as $\tilde{m}(r) = \frac{\Omega^2 r}{g(r)}$ and $m = \frac{\Omega^2 R}{g(R)}$ with g the gravity attraction at each radius:

$$g(r) = \frac{4\pi G \bar{\rho} r}{3} = \frac{4\pi G}{r^2} \int_0^r \rho(y) y^2 \, \mathrm{d}y. \tag{8}$$

Define the mass of the Earth reference model:

$$M = 4\pi \int_0^R \rho(y) y^2 \, \mathrm{d}y, \tag{9}$$

The parameter m, which is a measure of the ratio of centrifugal and gravitational forces, is small ($m \simeq 1/289$) and the f_n can be estimated at first, second, etc, order with respect to m, by integration of differential systems where the variable is r. Those relations were first established by Clairaut (1743) for the first order, Callandreau (1889) up to second order and Lanzano (1962) to third order. For the Earth the second order is necessary and sufficient. The equations are given in Kopal (1960), Lanzano (1982). Moritz (1990) gives the equations for other variables (ellipticity instead of f). We have verified the equations up to second order. Missprints and mistakes are reported in (Chambat, Ricard, Valette, 2010).

2 Equations to solve

2.1 First order

At the first order $f_4(r) \equiv 0$. Note:

$$y_1(r) = f_2(r) \quad y_2(r) = r\dot{f}_2(r) = r\dot{y}_1(r) \tag{10}$$

where a dot denotes $\mathrm{d}/\mathrm{d}r$. y_1, y_2 is the solution of:

$$\dot{y}_1 = \frac{y_2}{r} \tag{11}$$

$$\dot{y}_2 = 6(1-\gamma)\frac{y_1}{r} + (1-6\gamma)\frac{y_2}{r} \tag{12}$$

with the condition at the center:

$$y_2(0) = 0, \tag{13}$$

and at the surface:

$$(2y_1 + y_2)(R) = -\frac{5}{3}m. \tag{14}$$

2.2 Second order

At the second order:

$$f_2(r) = y_1(r) + z_1(r), \quad f_4(r) = z_3(r), \tag{15}$$

where y_1 is the first order solution calculated above and z_1, z_3 is the second order solution given in the following way. Define

$$z_2(r) = r\dot{z}_1(r), \quad z_4(r) = r\dot{z}_3(r). \tag{16}$$

Then the systems are the following.

a. Degree $n = 2$. The functions z_1, z_2 are the solution of:

$$\dot{z}_1 = \frac{z_2}{r} \tag{17}$$

$$\dot{z}_2 = 6(1-\gamma)\frac{z_1}{r} + (1-6\gamma)\frac{z_2}{r} + \frac{2}{7}\left\{9(2-\gamma)y_1 + (2-9\gamma)y_2\right\}\frac{y_2}{r} + 4\,\tilde{m}\,(1-\gamma)(y_1+y_2) \tag{18}$$

with the condition at the center:

$$z_2(0) = 0, \tag{19}$$

and at the surface:

$$(2z_1 + z_2)(R) = \frac{2}{7}\left\{6y_1^2 + 3y_1y_2 + y_2^2\right\}(R) + \frac{2}{3}\,m\,(5y_1 + y_2)(R). \tag{20}$$

b. Degree $n = 4$. The functions z_3, z_4 are the solution of:

$$\dot{z}_3 = \frac{z_4}{r} \tag{21}$$

$$\dot{z}_4 = (20-6\gamma)\frac{z_3}{r} + (1-6\gamma)\frac{z_4}{r} + \frac{18}{35}\left\{2y_2(2y_1+y_2) - 3\gamma(7y_1^2 + 6y_1y_2 + 3y_2^2\right\}\frac{1}{r} \tag{22}$$

with the condition at the center:

$$z_3(0) = \frac{27}{35}y_1^2(0), \tag{23}$$

and at the surface:

$$(4z_3 + z_4)(R) = \frac{18}{35}\left\{6y_1^2 + 5y_1y_2 + y_2^2\right\}(R). \tag{24}$$

3 Numerical integration

3.1 First order

Since the system is homogeneous (no second term) and linear the numerical integration is easy.

Close to the center r^{n-2} and r^{-n-3} are solutions. For $n = 2$ the only admissible solution is thus $(y_1, y_2) \simeq (cste, 0)$. Start at a radius $r = r_0 \ll R$ close to the center, with

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(r_0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{25}$$

Integrates the differential system from r_0 to R with a Runge-Kutta solver. Note (u_1, u_2) the result. The solution is then given by normalisation with the surface condition:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(r) = -\frac{5}{3}\frac{m}{(2u_1 + u_2)(R)}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(r). \tag{26}$$

Practical condensed notations can be used to note all this. Note $Y = (y_1, y_2)$ then equations (11)-(12), (13), (14) can be respectively written:

$$\dot{Y} = AY, \quad Y|_2(0) = 0, \quad CL(Y)(R) = CL_R \tag{27}$$

where $A(r)$ is a matrix, '$Y|_2$' means the second component of Y, 'CL' means the linear combination of Y components appearing in the left side of (14), and CL_R the constant on the right side of (14). Note $U = (u_1, u_2)$, then the normalisation reads:

$$Y(r) = CL_R \frac{U(r)}{CL(U)}. \tag{28}$$

3.2 Second order

Despite recommandation of Kopal (1960) and Moritz (1990), everybody seems to have used iteration to achieve the integration of both differential systems[1]. In fact that is not usefull since the systems are linear. The squared terms that appear can indeed be considered as products of first order terms that are already known, and not as products of first+second order terms.

3.2.1 Principles

Each system can be written as:

$$\dot{Z} = AZ + S, \quad Z(0) = CL_0, \quad CL(Y)(R) = CL_R \tag{29}$$

where Z is either (z_1, z_2) or (z_3, z_4) and '$Z|_2$' means the second component of Z. Since that system is heterogeneous the resolution proceeds in two integrations: one for a particular solution of the heterogeneous system and of the general solution of the homogenous system. The general solution thus reads:

$$Z = P + \alpha U + \beta V \tag{30}$$

[1] (17)-(20) for degree $n = 2$, (21)-(24) or $n = 4$.

where P is a particular solution, U and V are two independent solutions of the homogeneous system (fundamental system), α and β two constants to be determined.

Close to the center, r^{n-2} and r^{-n-3} are solutions of $\dot{Z} = AZ$. Choose U to be the r^{n-2} solution, then V is infinite at the center. That imposes $\beta = 0$. Thus:

$$Z = P + \alpha U. \tag{31}$$

The constant α is given by the surface condition $CL(P) + \alpha CL(U) = CL_R$.

At the center, for $n = 2$: $U \simeq (cste, 0)$ and the shape of the particular is not imposed, while for $n = 4$: $U \simeq cste(r^2, 2r^2) \simeq (0, 0)$ and the system imposes that $z_3 \simeq \frac{27}{35} y_1^2$.

Thus the practical method follows:

3.2.2 Practical

a. Particular solution. Start with

$$Z(r_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ for } n = 2 \text{ and } \quad Z(r_0) = \frac{27}{35} y_1^2(r_0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ for } n = 4. \tag{32}$$

Integrates $\dot{Z} = AZ + S$ from r_0 to R with a Runge-Kutta solver. That gives $P(r)$.

b. Homogeneous solution. Start with

$$Z(r_0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ for } n = 2 \text{ and } \quad Z(r_0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ for } n = 4. \tag{33}$$

Integrates $\dot{Z} = AZ$ from r_0 to R with a Runge-Kutta solver. That gives $U(r)$.

c. General solution. The general solution is then

$$Z(r) = P(r) + (CL_R - CL(P)) \frac{U(r)}{CL(U)}. \tag{34}$$

That might be better to numerically compute it as:

$$Z(r) = \left(P(r) - CL(P) \frac{U(r)}{CL(U)} \right) + CL_R \frac{U(r)}{CL(U)}. \tag{35}$$

For $n = 4$ the two terms in the parenthesis are numerically close each other. Since their relative difference is about 10^{-6} that is not a problem if one works with double precision numbers.

3.3 Potential

The external gravitationnal potential of the hydrostatic Earth can be written as:

$$\phi = \frac{GM}{r} \left(1 - J_2 \frac{a^2}{r^2} P_2(\cos\theta) - J_4 \frac{a^4}{r^4} P_4(\cos\theta) \right). \tag{36}$$

where J_2 and J_4 are non-dimensionnal coefficients, and where it must be emphasised that the length a in this expression is conventionnal and is usually chosen as the semi-major axis of the reference ellispoïd or of the hydrostatic model.

If we make the second choice we have the followings relation (Nakiboglu, 1982, verified). At the first order:

$$J_2 = -f_2 - m/3, \qquad J_4 = 0, \tag{37}$$

where, f_2 here means $f_2 = y_1(R)$. At the second order:

$$J_2 = -f_2 - m/3 - 11f_2^2/7 - mf_2/7, \qquad J_4 = -f_4 - 36f_2^2/35 - 6mf_2/7, \tag{38}$$

where, here :

$$f_2 = y_1(R) + z_1(R), f_4 = z_3(R). \tag{39}$$

The hydrostatic major semi-axis is:

$$a = R(1 - f_2/2 + 3f_4/8). \tag{40}$$

3.4 Numerical results

3.4.1 Choice of constants

I use a density model in a discretized form, matlab routine order 4 Runge-Kutta `ode45` to integrate, and the matlab spline interpolator `interp1` to give values between points. With relative and absolute tolerances of 10^{-12} and 10^{-16} in `ode45`, the program runs in several seconds on a simple MacBook. The relative error in the limit conditions is conditionned by the starting radius. With a radius of 100 m, the error is several 10^{-10}.

Notice that the result depends upon $G\rho$ only and not upon G and ρ independently. To see it is sufficient to write g as:

$$g(r) = \frac{GM}{R^2}\frac{\bar{\rho}(r)r}{\bar{\rho}(R)R}. \tag{41}$$

3.4.2 Results

Results given in (Chambat, Ricard, Valette, 2010) are stable.

4 References

- Callandreau, O., 1889, Ann. Obs. Paris, 19 E.
- Clairaut, A. C., 1743, Théorie de la figure de la Terre, Tirée des principes de l'hydrostatique, David Fils or Durand, Paris.
- Kopal, Z., 1960, Figures of equilibrium of celestial bodies, Univ. Wisconsin Press, Madison.
- Lanzano, P., 1962, A third-order theory for the equilibrium configuration of a rotating planet, Icarus, 1, 121.
- Lanzano, P., 1982, Deformations of an elastic Earth, Int. Geophys. Series, vol. 31, Academic Press, New York.
- Moritz, H., 1990, The Figure of the Earth, Wichmann, Karlsruhe.
- Nakiboglu, S. M., 1982, Hydrostatic theory of the Earth and its mechanical implications, Phys. Earth. Planet. Int., 28-302-311.