Relating Gravity, Density, Topography and State of Stress Inside a Planet

B. Valette
LGIT, IRD, Université de Savoie, F-73376 Le Bourget du Lac Cedex, France.
e-mail: Bernard.Valette@univ-savoie.fr

F. Chambat
ENS Lyon, LST, 46, Allée d'Italie, F-69364 Lyon Cedex 07, France.
e-mail: Frederic.Chambat@ens-lyon.fr

Abstract. Current interpretations of gravimetric and topographic data rely either on isostasy or on thin plate bending theory. Introducing a fluid rheology constitutes an alternative for global interpretation. In this paper, we present a method that enables to directly relate gravity to deviatoric stresses without any rheological assumption. The relation is obtained by perturbing the equilibrium equation and Poisson's equation around a static spherical configuration, and by introducing a set of suited variables. Namely, we consider the density variation over the equipotential surfaces and the height of interfaces above their corresponding equipotential surfaces. The Backus decomposition of second-order tensors in scalar potentials (Backus 1966) is also found to be very useful. Finally, we show that the method can provide a way to infer strength differences and crustal thickness in a way that generalizes the isostasy approach.

Keywords. Perturbation, topography, Clairaut's equation, gravity, stress, density.

1 Setting the Problem

The relation between the shape of planets and the equilibrium equation has been intensively studied in the hydrostatic context. This has yielded, since Clairaut's work, the classical studies on equilibrium figures. In other respects, the local gravimetric and topographic data are usually interpreted in the framework of isostasy or of plate bending theory. In global approaches, the gravity potential is commonly related to density and to discontinuity topographies through a first-order Eulerian perturbation. Moreover it has become usual to consider a Newtonian fluid rheology in order to relate topography to density variations and to interpret tomographic images in terms of density and gravity.

In this paper we adopt a starting point of view similar to that of Backus (1967) or Dahlen (1981) and refer to the ambient state of stress, without any rheological consideration. Thus we avoid to consider a physical process and to define an initial thermodynamical configuration with the complications it implies. Our aim is only to determine the relations that can be established with the ambient state of stress. More precisely, let us consider a planet occupying the domain V of the space refered to a co-rotating frame located at the centre of mass of the body. Our purpose is to explain how the gravity field of the planet can be written in terms of stress instead of density field, independantly of any rheological law. It consists in solving together Poisson's equation:

$$\Delta\varphi = 4\pi G\rho - 2\Omega^2, \quad \rho(x) = 0 \text{ for } x \in \mathbf{R^3} \setminus V, \tag{1}$$

with the equation of equilibrium:

$$\text{div}\sigma - \rho\text{grad}\varphi = 0, \quad \sigma(x) = 0 \text{ for } x \in \mathbf{R^3} \setminus V, \tag{2}$$

while satisfying the usual boundary conditions:

$$[\varphi] = 0, \quad \varphi(x) + (\Omega^2 x^2 - (\Omega\cdot x)^2)/2 \underset{\infty}{\rightarrow} 0, \tag{3}$$

$$[\text{grad}\varphi \cdot n] = 0, \tag{4}$$

$$[\sigma(n)] = 0, \tag{5}$$

where x, ρ, G, φ, $-\text{grad}\varphi$, Ω, σ denote the position-vector, the density, the gravitational constant, the gravity potential, the gravity vector, the (constant) rotation vector and the Cauchy stress tensor, respectively. $[\ \ ]$ denotes the jump across the closed interfaces Σ, including the outer boundary ∂V, oriented by the unit normal vector field n. In order to solve this system we use a perturbation method because: (a) Equation (2) depends non-linearly on ρ; (b) the shape is involved in the solutions of the equations; (c) planets have a quasi-spherical symmetry.

The paper is structured as follows. In section 2, we set up the shape perturbation formalism. Section 3 is devoted to the perturbation of Poisson's equation and to the generalization of Clairaut's equation by introducing non-hydrostatic variables. Section 4 is devoted to the perturbation of the equilibrium equation and to the expression of the non-hydrostatic variables as functions of the Backus potentials of the deviatoric stress tensor. Finally, in section 5 we outline an inversion scheme of gravity and topography models, considered as data, that relies on the global minimization of the strength difference. Minimizing the deviatoric stress was also considered by Dahlen (1981, 1982) as a possible interpretation of isostasy.

2 Perturbation Formalism

Let us begin by defining the reference hydrostatic spherical configuration as in Chambat & Valette (2001). To this purpose, we first consider a continuous set of surfaces S which interpolate the interfaces Σ from the centre of mass to the boundary ∂V. Secondly, let us define the mean radius r of S as the angular average of the distance of the centre of mass to the points of S. Let us denote by $r = b$ the mean radius of ∂V. We can now define the mean density $\rho(r)$ as the angular average of ρ over S. The potential φ and the pressure p are finally deduced through equations (1-5) with $\sigma = -p\mathrm{I}_d$, $\Omega = 0$ and $V = B(0, b)$. Now, the real configuration must be related to the reference one by introducing a continuous evolution. The physical parameters can be derived from the reference ones through a Taylor expansion which defines the perturbations to the different orders. The deformation of the domain is parameterized by a scalar t ranging from 0, for the reference domain $V_0 = B(0, b)$, to 1 for the real domain V, and which can be thought of as a virtual time. More precisely, let us consider a mapping: $\forall(a, t) \in V_0 \times [0, 1]$, $(a, t) \to x(a, t) \in V_t$ with $\forall a \in V_0, x(a, 0) = a$, $x(a, 1) = x \in V_1 = V$. For any regular tensor field T we consider a mapping: $\forall(a, t) \in V_0 \times [0, 1]$, $(a, t) \to T(x(a, t), t)$ with $T(a, 0)$ corresponding to the reference field in V_0 and $T(x, 1)$ to the real one in V. The n^{th} order Lagrangian displacement is defined as:

$$\xi_n(a) = \left. \frac{\mathrm{d}^n}{\mathrm{d}t^n} x(a, t) \right|_{t=0},$$

and the Eulerian, respectively Lagrangian, n^{th} order perturbation of T as:

$$\delta_{ne} T(a) = \left. \frac{\partial^n}{\partial t^n} T(x(a, t), t) \right|_{t=0},$$

$$\delta_{n\ell} T(a) = \left. \frac{\mathrm{d}^n}{\mathrm{d}t^n} T(x(a, t), t) \right|_{t=0}.$$

Thus, $\delta_{ne} x = 0$ and $\delta_{n\ell} x = \xi_n$. Defining ξ, $\delta_e T$ and $\delta_\ell T$ respectively by:

$$x(a, 1) = a + \xi(a), \quad T(a, 1) = T(a, 0) + \delta_e T(a),$$

$$T(x(a, 1), 1) = T(a, 0) + \delta_\ell T(a),$$

a Taylor expansion of order N yields:

$$\xi(a) = \sum_{n=1}^{N} \frac{\xi_n}{n!}(a),$$

$$\delta_e T(a) = \sum_{n=1}^{N} \frac{1}{n!} \delta_{ne} T(a),$$

$$\delta_\ell T(a) = \sum_{n=1}^{N} \frac{1}{n!} \delta_{n\ell} T(a).$$

From the definition, it is clear that the Eulerian perturbations commutate with the spatial differentiations. Consider now a scalar field f, a vector field u and a symmetric second order tensor field T. The following usual first-order relations hold:

$$\delta_\ell f = \delta_e f + \mathrm{grad} f \cdot \xi, \tag{6}$$

$$\delta_\ell u = \delta_e u + \nabla u(\xi), \tag{7}$$

$$\delta_\ell(\mathrm{div} T) = \mathrm{div}(\delta_\ell T) - \nabla T : \nabla \xi, \tag{8}$$

where:

$$(\nabla T : \nabla \xi)^j = \nabla_k T^{ij} \nabla_i \xi^k. \tag{9}$$

Finally, we impose that $\delta\Omega^2 = \delta_1\Omega^2 = \Omega^2$, and that the deformation is purely radial, *i. e.*, $\xi = h_\xi\, e_r$ where e_r is the unit radial vector.

3 Generalizing Clairaut's Equation

The purpose of this section is to explain how Poisson's equation can be solved in a way which allows to generalize Clairaut's equation. This is done by introducing new variables which permit to separate topographies from equipotential heights and to identify non-hydrostatic density repartition. The classical way to solve Poisson's equation in a quasi-spherical geometry is to use a perturbation approach between the non-rotating mean model and the aspherical model rotating with angular velocity Ω, and to consider the Eulerian perturbation of potential $\delta_e\varphi$ and of density $\delta_e\rho$. This leads to:

$$\Delta\delta_e\varphi = 4\pi G \delta_e\rho - 2\Omega^2, \tag{10}$$

with the following interface conditions:

$$[\delta_e\varphi] = 0, \quad [\mathrm{grad}(\delta_e\varphi) + 4\pi G\rho\xi] \cdot e_r = 0. \tag{11}$$

Expanding $\delta_e\varphi$ and $\delta_e\rho$ in spherical harmonics yields for each l, m:

$$\left\{\partial_r^2 + \frac{2}{r}\partial_r - \frac{l(l+1)}{r^2}\right\}\delta_e\varphi = 4\pi G\delta_e\rho - 2\Omega^2\delta_l^0, \qquad (12)$$

with boundary conditions:

$$[\delta_e\varphi] = 0, \quad [\partial_r\delta_e\varphi + 4\pi G\rho h_\xi] = 0, \qquad (13)$$

where we have dropped the indices l and m in the coefficients. The spherical harmonics used here are normalized as

$$\int_0^{2\pi}\int_0^{\pi} Y_l^m(\theta,\lambda)\bar{Y}_{l'}^{m'}(\theta,\lambda)\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\lambda = 4\pi\delta_l^{l'}\delta_m^{m'}$$

where θ and λ denote colatitude and longitude, respectively. For instance:

$$Y_0^0(\theta,\lambda) = 1, \quad Y_2^0(\theta,\lambda) = \sqrt{5}(3\cos^2\theta - 1)/2.$$

3.1 Introducing New Variables

For each degree $l \neq 0$, let us now consider the variables:

$$h_\varphi = \frac{\delta_e\varphi}{g},\ \delta_\varphi\rho = \delta_e\rho + h_\varphi\partial_r\rho,\ h = h_\xi - h_\varphi, \qquad (14)$$

where g denotes the (negative) radial gravity in the reference state and satisfies:

$$\partial_r g + 2g/r + 4\pi G\rho = 0. \qquad (15)$$

h_φ is the first order equipotential height above the sphere of radius r and h is the height above the equipotential surface. $\delta_\varphi\rho$ represents the lateral variations of density over the associated equipotential surface. Note that applying $\delta_\varphi = \delta_e + h_\varphi\,\partial_r$ corresponds to perturbing to the first order while following the equipotential surfaces and that, for $r = b$, h_φ corresponds to the geoid height and h to the altitude. It is also useful to define:

$$k = \sqrt{(l-1)(l+2)}, \qquad (16)$$

$$\gamma = \frac{\rho}{\rho_2} = -\frac{4\pi G\rho r}{3g} = \frac{\partial_r(gr^2)}{3gr}. \qquad (17)$$

γ is the ratio of the reference density ρ at radius r to the mean density $\rho_2 = 3\int_0^r \rho(s)s^2\,\mathrm{d}s/r^3$ inside the sphere of radius r.
Using these variables and taking (15) into account, equations (12,13) can be rewritten as:

$$\partial_r^2 h_\varphi - \frac{2}{r}(1-3\gamma)\,\partial_r h_\varphi - \frac{k^2}{r^2}h_\varphi = \frac{4\pi G}{g}\delta_\varphi\rho, \qquad (18)$$

or

$$\partial_r\begin{pmatrix} h_\varphi \\ r\partial_r h_\varphi\end{pmatrix} = \frac{1}{r}\begin{pmatrix} 0 & 1 \\ k^2 & 3(1-2\gamma)\end{pmatrix}\begin{pmatrix} h_\varphi \\ r\partial_r h_\varphi\end{pmatrix} - 3\gamma\begin{pmatrix} 0 \\ \delta_\varphi\rho/\rho\end{pmatrix}, \qquad (19)$$

with:

$$[h_\varphi] = 0, \quad [\partial_r h_\varphi] = 3\,[\gamma]\,h/r, \qquad (20)$$

$$b\partial_r h_\varphi(b) + (l-1)h_\varphi(b) = 3\gamma h(b) + \frac{\sqrt{5}}{3}\frac{\Omega^2 b^2}{g(b)}\delta_l^2\delta_m^0, \qquad (21)$$

$$h_\varphi \underset{r\to 0}{\simeq} \mathrm{cst}\; r^{l-1}. \qquad (22)$$

The location of the frame origin at the centre of mass yields the additional condition for $l = 1$:

$$h_\varphi(b) = 0. \qquad (23)$$

Condition (21) is derived from (13) by noting that, in virtue of (3) and (12), $\delta_e\varphi$ can be expressed for $r > b$ as:

$$\delta_e\varphi = \frac{\mathrm{cst}}{r^{l+1}} + \frac{\Omega^2 r^2}{3\sqrt{5}}\delta_l^2\delta_m^0,$$

which yields:

$$\partial_r\delta_e\varphi = -(l+1)\frac{\delta_e\varphi}{r} + \frac{\sqrt{5}}{3}\Omega^2 r\,\delta_l^2\delta_m^0.$$

Condition (22) stems from the behaviour of the degree l component of the regular scalar function $\delta_e\varphi$.
Let us now consider the homogeneous system corresponding to (19):

$$\partial_r\begin{pmatrix} x \\ r\partial_r x\end{pmatrix} = \frac{1}{r}\begin{pmatrix} 0 & 1 \\ k^2 & 3(1-2\gamma)\end{pmatrix}\begin{pmatrix} x \\ r\partial_r x\end{pmatrix}$$

$$[h_\varphi] = 0, \quad [\partial_r h_\varphi] = 0. \qquad (24)$$

Noting that $\gamma(0) = 1$ (see 17), it can be shown that all the solutions of (24) behave like $1/r^{l+2}$ in the vicinity of the centre, except a line of solutions which are proportionnal to r^{l-1}. Let h_1 be such a solution, defined to a multiplicative constant, let h_2 be the solution defined by $(h_2, r\partial_r h_2)(b) = (1, -l+1)$, and let $F(r)$ be the matrix defined as:

$$F = \begin{pmatrix} h_1 & h_2 \\ r\partial_r h_1 & r\partial_r h_2\end{pmatrix}. \qquad (25)$$

From (24), we can deduce that:

$$\partial_r \det(F) = 3\det(F)(1-\gamma)/r, \quad (26)$$

and, taking (17) into account, that:

$$\det(F(r)) = \det(F(b))\frac{bg^2(b)}{rg^2(r)}. \quad (27)$$

Following Poincaré (1902, p. 84), now we will show that h_1 and h_2 remains close to r^{l-1} and to $1/r^{l+2}$ respectively and that F(r) is a fundamental matrix of (24) for $l > 1$.

3.2 Setting Bounds on h_1 and h_2

Let us assume that, for any r, $0 \le \gamma(r) \le 1$, i.e., that $0 \le \rho(r) \le \rho_2(r)$ or that ρ_2 is decreasing with r. Note that this hypothesis is weaker than the one of a decreasing density. Let γ_0 be the minimum of $\gamma(r)$ over [0,b] and define q as:

$$q = \frac{1}{2}\left(3(1-2\gamma_0) + \sqrt{9(1-2\gamma_0)^2 + 4k^2}\right). \quad (28)$$

Under the above hypothesis, h_1 and h_2 satisfy for any r $\in [0, b]$:

$$l - 1 \le r\frac{\partial_r h_1}{h_1}(r) \le q \le l + 2, \quad (29)$$

$$-(l+2) \le r\frac{\partial_r h_2}{h_2}(r) \le -(l-1), \quad (30)$$

and thus by integration:

$$\left(\frac{r}{b}\right)^{l+2} \le \left(\frac{r}{b}\right)^{q} \le \frac{h_1(r)}{h_1(b)} \le \left(\frac{r}{b}\right)^{l-1}, \quad (31)$$

$$\left(\frac{b}{r}\right)^{l-1} \le h_2(r) \le \left(\frac{b}{r}\right)^{l+2}. \quad (32)$$

For $l = 1$ ($k = 0$), the proof of (29) and (31) is straightforward, since in this case, h_1 is constant and $h_2 = 1$ on the interval $[0, b]$. For $l \ge 2$, let us define after Poincaré:

$$\alpha_1(r) = r\frac{\partial_r h_1}{h_1}(r), \quad \alpha_2(r) = r\frac{\partial_r h_2}{h_2}(r). \quad (33)$$

The definitions are a posteriori justified by the fact that α_1 and α_2 remain finite, i.e., that h_1 and h_2 do not vanish except for $r = 0$. α_1 and α_2 obey the differential equation:

$$\partial_r \alpha = -\left(k^2 + 3(1-2\gamma)\alpha - \alpha^2\right)/r, \quad (34)$$

which can be reformulated as:

$$\partial_r \alpha = -(\alpha - \alpha_+)(\alpha - \alpha_-)/r, \quad (35)$$

with:

$$\alpha_\pm = \frac{1}{2}\left(3(1-2\gamma) \pm \sqrt{9(1-2\gamma)^2 + 4k^2}\right). \quad (36)$$

Differentiating (36) with respect to γ yields:

$$\partial_\gamma \alpha_+ = \frac{-6\alpha_+}{\sqrt{9(1-2\gamma)^2 + 4k^2}}. \quad (37)$$

Since $\alpha_+ \ge 0$, α_+ is a decreasing function of γ, and thus for any $r \in [0\,,\, b]$:

$$0 < \alpha_+(\gamma = 1) = l - 1 \le \alpha_+(r) \le$$

$$\le \alpha_+(\gamma_0) = q \le \alpha_+(\gamma = 0) = l + 2. \quad (38)$$

The relation $\alpha_-\alpha_+ = -(l-1)(l+2)$ shows that α_- is a negative decreasing function of γ and that:

$$-(l+2) \le \alpha_-(r) \le -(l-1) < 0. \quad (39)$$

At the centre, $\gamma = 1$ so that $\alpha_+ = l - 1, \alpha_- = -(l+2)$ and $\alpha_1(0) = l - 1$. Noting that $\partial_r \alpha \ge 0$ for $\alpha \in [\alpha_-, \alpha_+]$ and $\partial_r \alpha < 0$ outside, we conclude that $x(r)$ remains in the interval $[l-1, q]$, that is (29). Noting that (see 25):

$$\det(F(b)) = -h_1(b)(l - 1 + \alpha_1(b)), \quad (40)$$

we can conclude from (27) and (29) that F is a fundamental matrix of (24), i.e., $\det(F) \ne 0$, and consequently that $\alpha_2(0) = -(l+2)$. Remarking once again that $\partial_r \alpha > 0$ for $\alpha > -(l-1)$ and $\partial_r \alpha < 0$ for $\alpha < -(l+2)$ yields (30), which ends up the proof.
In order to make completely clear the definition of h_1, inequality (29) shows that for $l > 1$ we can normalize it as:

$$b\partial_r h_1(b) + (l-1)h_1(b) = 1. \quad (41)$$

In this case, (40) yields $\det(F(b)) = -1$ and it follows from (27) that:

$$\det(F(r)) = -\frac{bg^2(b)}{rg^2(r)}, \quad (42)$$

$$F^{-1}(r) = -\frac{rg^2(r)}{bg^2(b)}\begin{pmatrix} r\partial_r h_2 & -h_2 \\ -r\partial_r h_1 & h_1 \end{pmatrix}. \quad (43)$$

3.3 Clairaut's Equation

Under the hydrostatic hypothesis, the level surfaces of density, potential and pressure coincide. In addition these level surfaces include the interfaces. Therefore $\delta_\varphi \rho$ and h, as defined by (14), identically vanish. Equations (19-23) then reduce to the homogeneous system (24) with the boundary conditions (22) and:

$$b\partial_r h_\varphi(b) + (l-1)h_\varphi(b) = \frac{\sqrt{5}}{3}\frac{\Omega^2 b^2}{g(b)}\delta_l^2\delta_m^0, \qquad (44)$$

with the additional condition (23) for $l = 1$. This set of equations is an alternative form of Clairaut's differential equation. Paragraph 3.2 shows that it can only be fulfilled for $l = 2$ and $m = 0$ and thus that, as it is well known (see for instance Jeffreys (1976)), the solution only contains the degree 2 order 0 term:

$$h_\varphi(r) = \frac{\sqrt{5}}{3}\frac{\Omega^2 b^2}{g(b)}\delta_l^2\delta_m^0 h_1(r). \qquad (45)$$

In the general case of a non-hydrostatic repartition of density we must solve the system (19-23) with the use of h_1 and h_2.

3.4 h_φ as a Function of Topographies and Non-hydrostatic Density Repartition

Let us now show that for $l > 1$:

$$h_\varphi(b) = -\frac{4\pi G}{bg^2(b)}\left\{\int_0^b g\, h_1\, \delta_\varphi\rho(r)\, r^2 \mathrm{d}r \right. \qquad (46)$$

$$\left. - \sum_{r_\Sigma \le b} r_\Sigma^2 g\,[\rho]\, h\, h_1(r_\Sigma)\right\} + \frac{\sqrt{5}}{3}\frac{\Omega^2 b^2}{g(b)}\delta_l^2\delta_m^0 h_1(b),$$

and more generally that:

$$h_\varphi(r) = -\frac{4\pi G}{bg^2(b)}\left\{h_1(r)\left(\int_r^b g\, h_2\, \delta_\varphi\rho(s)\, s^2 ds \right.\right.$$

$$\left. - \sum_{r<r_\Sigma \le b} r_\Sigma^2 g\,[\rho]\, h\, h_2(r_\Sigma)\right)$$

$$+h_2(r)\left(\int_0^r g\, h_1\, \delta_\varphi\rho(s)\, s^2 ds \right. \qquad (47)$$

$$\left.\left. - \sum_{r_\Sigma \le r} r_\Sigma^2 g\,[\rho]\, h\, h_1(r_\Sigma)\right)\right\} + \frac{\sqrt{5}}{3}\frac{\Omega^2 b^2}{g(b)}\delta_l^2\delta_m^0 h_1(r),$$

where r_Σ denotes the mean radius of interfaces, including the external boundary.
In order to prove these relations, let us first define:

$$X(r) = \begin{pmatrix} h_\varphi(r) \\ r\partial_r h_\varphi(r) \end{pmatrix},\ S(r) = \begin{pmatrix} 0 \\ -3\gamma\delta_\varphi\rho/\rho \end{pmatrix}.$$

A solution $X(r)$ of (19, 20, 22) can be written as:

$$X(r) = F(r)\left\{X_0 + \sum_{r_\Sigma < r} F^{-1}(r_\Sigma)[X(r_\Sigma)] \right.$$

$$\left. + \int_0^r F^{-1}(s)S(s)\mathrm{d}s\right\}, \qquad (48)$$

where the constant vector X_0 can be taken in the form:

$$X_0 = \begin{pmatrix} (4\pi G/bg^2(b))\, c \\ 0 \end{pmatrix},$$

and where:

$$[X(r_\Sigma)] = \begin{pmatrix} 0 \\ 3[\gamma(r_\Sigma)]h(r_\Sigma) \end{pmatrix}.$$

Applying $F^{-1}(r)$ (see 43) to equation (48) and setting $r = b$ in the resulting second component yields:

$$b^2 g^2(h_\varphi\partial_r h_1 - h_1\partial_r h_\varphi)(b) =$$

$$-\sum_{r_\Sigma < b} 3r_\Sigma g^2[\gamma]hh_1(r_\Sigma) + 3\int_0^b g^2\gamma h_1\frac{\delta_\varphi\rho}{\rho}(s)\, s\, \mathrm{d}s.$$

Taking boundary condition (21) into account in the left hand side of this equation leads to:

$$b^2 g^2(h_\varphi\partial_r h_1 - h_1\partial_r h_\varphi)(b) =$$

$$bg^2\left(h_\varphi(b) - 3\gamma h h_1(b) - \frac{\sqrt{5}}{3}\frac{\Omega^2 b^2}{g(b)}\delta_l^2\delta_m^0 h_1(b)\right),$$

and finally to (46) by making use of (17). In order to prove (47), let us start from the first component of equation (48) which can be written as:

$$h_\varphi(r) = \frac{4\pi G}{bg^2(b)}\left\{h_1(r)\left(\int_r^b g\, h_2\, \delta_\varphi\rho(s)\, s^2 ds \right.\right.$$

$$\left. - \sum_{r_\Sigma < r} r_\Sigma^2 g\,[\rho]\, h\, h_2(r_\Sigma) + c\right)$$

$$+h_2(r)\left(-\int_0^r g\, h_1\, \delta_\varphi\rho(s)\, s^2 ds \right. \qquad (49)$$

$$\left.\left. + \sum_{r_\Sigma < r} r_\Sigma^2 g\,[\rho]\, h\, h_1(r_\Sigma)\right)\right\}.$$

Putting $r = b$ in this equation and substituting expression (46) of $h_\varphi(b)$ yields the value of the constant:

$$c = \sum_{r_\Sigma \le b} r_\Sigma^2 g\,[\rho]\, h\, h_2(r_\Sigma)$$

$$- \int_0^b g\, h_2\, \delta_\varphi \rho(s)\, s^2 ds + \frac{bg^2(b)}{4\pi G} \frac{\sqrt{5}}{3} \frac{\Omega^2 b^2}{g(b)} \delta_l^2 \delta_m^0 .$$

Finally, setting the latter value in (49) leads to (47).

3.5 The Degree $l = 1$

The degree one needs a special treatment since, in this case, the function h_1 and h_2 are constant and F is no longer a fundamental matrix. Besides, a constant degree one h_φ corresponds to a translation of the body and does not affect either the source terms (controlled by $\delta_\varphi \rho$) or the boundary conditions (related to h). Therefore this degree would be undetermined if the centre of mass was not fixed by condition (23).

Taking advantage of $k^2 = 0$, (18) reduces to:

$$\partial_r (r \partial_r h_\varphi) = 3r \partial_r h_\varphi (1 - 2\gamma)/r - 3\gamma \delta_\varphi \rho / \rho \quad (50)$$

which, with the conditions (20, 22), leads to:

$$r \partial_r h_\varphi(r) = \frac{4\pi G}{r g^2(r)} \left\{ \int_0^r g\, \delta_\varphi \rho(s)\, s^2 \mathrm{d}s - \sum_{r_\Sigma < r} r_\Sigma^2 g\, [\rho]\, h(r_\Sigma) \right\} . \quad (51)$$

Now, condition (21) at $r = b$ implies the following constraint on $\delta_\varphi \rho$ and h:

$$\int_0^b g\, \delta_\varphi \rho(s)\, s^2 \mathrm{d}s - \sum_{r_\Sigma \le b} r_\Sigma^2 g\, [\rho]\, h(r_\Sigma) = 0. \quad (52)$$

Furthermore, taking (22) into account, (51) yields:

$$h_\varphi(r) = 4\pi G \left\{ \int_0^r \frac{1}{s^2 g^2(s)} \int_0^s g \delta_\varphi \rho(t)\, t^2 \mathrm{d}t\, \mathrm{d}s - \sum_{r_\Sigma < r} r_\Sigma^2 g[\rho]\, h(r_\Sigma) \int_{r_\Sigma}^r \frac{\mathrm{d}s}{s^2 g^2(s)} + c_1 \right\} . \quad (53)$$

The constant c_1 can be determined by the additional condition (23). This yields:

$$c_1 = - \int_0^b \frac{1}{s^2 g^2(s)} \int_0^s g\, \delta_\varphi \rho(t)\, t^2 \mathrm{d}t\, \mathrm{d}s + \sum_{r_\Sigma < b} r_\Sigma^2 g\, [\rho]\, h(r_\Sigma) \int_{r_\Sigma}^b \frac{\mathrm{d}s}{s^2 g^2(s)} .$$

Thus (53) becomes:

$$h_\varphi(r) = 4\pi G \left\{ - \int_r^b \frac{1}{s^2 g^2(s)} \int_0^s g \delta_\varphi \rho(t)\, t^2 \mathrm{d}t\, \mathrm{d}s + \sum_{r_\Sigma < r} r_\Sigma^2 g[\rho]\, h(r_\Sigma) \int_r^b \frac{\mathrm{d}s}{s^2 g^2(s)} + \sum_{r \le r_\Sigma < b} r_\Sigma^2 g[\rho]\, h(r_\Sigma) \int_{r_\Sigma}^b \frac{\mathrm{d}s}{s^2 g^2(s)} \right\} .$$

An integration by part of the first integral yields an expression of $h_\varphi(r)$ similar to (47) provided that we set, for $l = 1$, $h_1 = 1$ and $h_2 = bg^2(b) \int_r^b \mathrm{d}s/(s^2 g^2(s))$. Note that since the condition $h_\varphi(b) = 0$ is conventional, only the compatibility condition (52) is relevant for this degree. Finally, (46) can formally be kept for $l = 1$ with $h_1 = 1$ and $h_\varphi(b) = 0$, since it is equivalent to (52).

4 Taking Stress Into Account

Let us now turn to the equilibrium equation (2). Note that in the spherical reference configuration:

$$\sigma = -p \mathrm{I}_d, \quad \mathrm{grad} p = -\rho \mathrm{grad} \varphi, \quad [p] = 0. \quad (54)$$

Using (8) the Lagrangian perturbation of (2) yields:

$$\mathrm{div} \delta_l \sigma - \nabla \sigma : \nabla \xi - \rho \delta_l \mathrm{grad} \varphi - \mathrm{grad} \varphi \delta_l \rho = 0. \quad (55)$$

Relations (6, 7, 9) respectively imply that:

$$\delta_l \rho = \delta_\varphi \rho + (h_\xi - h_\varphi)\, \partial_r \rho = \delta_\varphi \rho + h\, \partial_r \rho,$$

$$\delta_l\, \mathrm{grad} \varphi = \mathrm{grad}(g h_\varphi) + \nabla \nabla (\varphi)(\xi),$$

$$\nabla \sigma : \nabla \xi = \rho \nabla \xi^* (\mathrm{grad} \varphi),$$

where the star denotes the adjoint with respect to the usual $\mathbf{R}^3$ scalar product. Substituting these three relations into (55) finally yields:

$$\mathrm{div} \delta_\varphi \sigma + g \delta_\varphi \rho\, e_r = 0 \text{ with } \delta_\varphi \sigma = \delta_l \sigma + \rho g h \mathrm{I}_d . \quad (56)$$

The boundary conditions are directly obtained upon perturbing (5):

$$[\delta_l \sigma](e_r) = -[\sigma(\delta_l n)] = p[(\delta_l n] = 0. \quad (57)$$

It is useful to apply the Backus scalar potential representation (Backus 1966) to $\delta_l \sigma$. This representation generalizes to second-order tensors the usual representation of vector fields in radial, poloidal and torroidal potentials with respect to the sphere and leads to local relations. In addition, expanding the potentials in spherical harmonics constitutes an alternative to the use of generalized spherical harmonics. Let us first recall the Backus representation in the case of symmetric tensors.

4.1 Backus Representation of Real Second-order Symmetric Tensor Fields

Let T be a regular real valued second-order symmetric tensor field. There exists 6 uniquely determined real potential fields P, Q, R, L, M, N such that:

$$Q_0 = R_0 = M_0 = N_0 = M_1 = N_1 = 0, \qquad (58)$$

and:

$$\begin{aligned} T = & P e_r \otimes e_r \\ & + e_r \otimes (\mathrm{grad}_T(rQ) + \mathrm{grad}_T(rR) \times e_r) \\ & + (\mathrm{grad}_T(rQ) + \mathrm{grad}_T(rR) \times e_r) \otimes e_r \\ & + \mathrm{P}_T . \left\{ \left(L - r^2 \Delta_T M\right) \right. \\ & \left. + 2r^2 \left(\mathrm{H}_T(M) + \mathrm{H}'_T(N)\right) \right\} . \mathrm{P}_T, \end{aligned} \qquad (59)$$

where the indices $_0$ and $_1$ in (58) refer to the spherical harmonic degrees, where the index $_T$ in (59) refers to the sphere of radius r, and where $\times$ denotes the vectorial product. The projector P_T and the tangential gradient grad_T are defined as:

$$\mathrm{P}_T = \mathrm{I}_d - e_r \otimes e_r, \ \mathrm{grad}_T = \mathrm{grad} - e_r \, e_r \cdot \mathrm{grad},$$

and the differential operators over the sphere of radius r, H_T and Δ_T, are related to the covariant derivative ∇_T by:

$$\mathrm{H}_T = \nabla_T \nabla_T, \quad \Delta_T = \nabla_T \cdot \nabla_T.$$

The differential operator H'_T is defined from H_T by the relation:

$$2\mathrm{H}'_T(N)(u_T) = (\mathrm{H}_T(N) u_T) \times e_r - \mathrm{H}_T(N)(u_T \times e_r),$$

which holds for any regular scalar field N and any regular vector field u_T tangent to the sphere.
Moreover, it can be verified that (Backus, 1966, 1967):

$$\mathrm{tr}(T) = P + 2L, \qquad (60)$$

$$T(e_r) = P e_r + \mathrm{grad}_T(rQ) + \mathrm{grad}_T(rR) \times e_r, \qquad (61)$$

$$\mathrm{div} T = U e_r + \mathrm{grad}_T(rV) + \mathrm{grad}_T(rW) \times e_r, \qquad (62)$$

with:

$$U = \left\{ r\partial_r P + 2P + r^2 \Delta_T Q - 2L \right\} / r, \qquad (63)$$

$$V = \left\{ r\partial_r Q + 3Q + (r^2 \Delta_T + 2) M + L \right\} / r, \qquad (64)$$

$$W = \left\{ r\partial_r R + 3R + (r^2 \Delta_T + 2) N \right\} / r. \qquad (65)$$

Note finally that e_r is an eigenvector of T if and only if $R = Q = 0$ and that T is transversely isotropic with respect to e_r if and only if $R = Q = M = N = 0$. In the latter case: $T = P e_r \otimes e_r + L \mathrm{P}_T$

4.2 Expression of h and $\delta_\varphi \rho$ as Functions of Stress Potentials

Using relation (62) for $T = \delta_l \sigma$ and identifying the radial, poloidal and toroidal components in the Lagrangian perturbation of the equilibrium equation (56) results in:

$$\delta_\varphi \rho = -(U + \partial_r(\rho g h))/g, \qquad (66)$$

$$\rho g h = -rV, \qquad (67)$$

$$W = 0. \qquad (68)$$

Substituting (67) into (66) and taking (63-65) into account yields:

$$\begin{aligned} \delta_\varphi \rho = & \left\{ \partial_r \left(r^2 (L - P + (r^2 \Delta_T + 2) M \right. \right. \\ & \left. \left. + \partial_r(rQ)) \right) - r(r^2 \Delta_T + 2)(Q + 2M) \right\} / g r^2, \end{aligned} \qquad (69)$$

$$h = -(L + (r^2 \Delta_T + 2) M + r\partial_r Q + 3Q)/\rho g, \qquad (70)$$

$$r^3 R = -(r^2 \Delta_T + 2) \int_0^r s^2 N \, \mathrm{d}s. \qquad (71)$$

Expression (61) shows that boundary conditions (57) can be rewritten as:

$$[P] = [Q] = [R] = 0. \qquad (72)$$

Furthermore, it can be deduced from (70, 72) that:

$$[P - L - (r^2 \Delta_T + 2) M - \partial_r(rQ)] = [\rho] \, g \, h, \qquad (73)$$

at each interface. Finally, a spherical harmonic expansion yields expressions of $\delta_\varphi \rho$, h, and R similar to (69-71) in which the operator $(r^2 \Delta_T + 2)$ is replaced by $-k^2$.

4.3 Expression of $h_\varphi(b)$ as a Function of Stress Potentials

Let us substitute the expression of the harmonic coefficient $\delta_\varphi \rho$ derived from (69) into (46). Integrating by parts and taking boundary conditions (73) into account first yields:

$$\begin{aligned} h_\varphi(b) = & \frac{4\pi G}{b g^2(b)} \int_0^b \left\{ -k^2 (Q + 2M) h_1 \right. \\ & \left. + (L - P - k^2 M) r \partial_r h_1 + \partial_r(rQ) r \partial_r h_1 \right\} (r) \, r \mathrm{d}r \end{aligned}$$

$$+\frac{\sqrt{5}}{3}\frac{\Omega^2 b^2}{g(b)}\delta_l^2\delta_m^0 h_1(b). \quad (74)$$

Upon integrating by parts the third term of the previous integral, then taking (72) into account and noting that $(h_1, r\partial_r h_1)$ verifies (24), we finally obtain:

$$h_\varphi(b) = \frac{4\pi G}{bg^2(b)}\int_0^b \{-2k^2(M+Q)h_1$$
$$+\left(L - P - k^2M - 2(2-3\gamma)Q\right) r\partial_r h_1\} (r)\, r\mathrm{d}r$$
$$+\frac{\sqrt{5}}{3}\frac{\Omega^2 b^2}{g(b)}\delta_l^2\delta_m^0 h_1(b). \quad (75)$$

The topographies h are not directly involved in the expression. They only intervene through (73). Note also that (75) identically vanishes for $l = 1$. This is due to the fact that the integration of equation (56) over the reference domain also yields the three equations (52) corresponding to the order $m = 0, \pm 1$.

4.4 Taking Strength Lines Into Account

As far as we know, the problem of determining the different kind of strength line patterns - related to the principal stresses - which fulfil the boundary conditions, *i.e.*, normal to any interface where a fluid is involved, is an open problem. Let us assume that the strength lines are quasi-radial or quasi-spherical. We guess that this assumption is very likely even in the context of a convective process. This implies that the rotation R mapping the local spherical frame onto the principal stresses frame can be formally written to the first order as $R = \mathrm{I}_d + A$ where A is an antisymmetric operator, and that:

$$(\mathrm{I}_d + A).(-p\mathrm{I}_d + \delta_l\sigma).(\mathrm{I}_d - A) = -p\mathrm{I}_d + \delta_l\sigma,$$

correct to first order. Therefore e_r is an eigen-vector field of the field $\delta_l\sigma$ and, according to 4.1 and (71) the stress potentials Q, R and N vanish. As a consequence (75) reduces to:

$$h_\varphi(b) = \frac{4\pi G}{bg^2(b)}\int_0^b \{(L - P - k^2M)r\partial_r h_1(r) -$$
$$-2k^2Mh_1\} (r)\, r\mathrm{d}r + \frac{\sqrt{5}}{3}\frac{\Omega^2 b^2}{g(b)}\delta_l^2\delta_m^0 h_1(b), \quad (76)$$

with:

$$\delta_\varphi\rho = \left\{\partial_r\left(r^2(L - P - k^2M)\right) + 2k^2Mr\right\}/gr^2,$$
$$h = (k^2M - L)/\rho g, \quad (77)$$

and at the interfaces:

$$[L - P - k^2M](r_\Sigma) = -[\rho]g\, h(r_\Sigma). \quad (78)$$

5 Inferring Strength Differences

Let us outline how we can get some inference on stress differences in the Earth and extrapolate available information on the crustal thickness. For the sake of simplicity we will now assume that $M = 0$. In this case the only parameter is the stress difference $d = L - P$ between the averaged quasi-horizontal stress and the quasi-vertical one. Equations (76-78) then simplify (omitting the degree two Clairaut term) to:

$$\delta_\varphi\rho = \partial_r(r^2 d(r))/gr^2,\ [d](r_\Sigma) = -[\rho]gh(r_\Sigma), \quad (79)$$

$$h_\varphi(b) = \frac{4\pi G}{bg^2(b)}\int_0^b d(r)r^2\partial_r h_1(r)\ \mathrm{d}r. \quad (80)$$

Using these last two equations $d(r)$ can be identified from geoid height $h_\varphi(b)$ and topography $h(b)$, both considered as data, through a functional least squares approach. Additional information can be obtained from models of discontinuity height $h(r_\Sigma)$ or density $\delta_\varphi\rho$ through equations (79). The regularization imposes a global minimization of $d(r)$ and corresponds to a simple mechanical criterion. Note that since the kernel $r^2\partial_r h_1(r)$ is close to r^l, the information is localized in the lithosphere, hence only the Moho topography is involved. Seismic models of Moho topography are now available and are reliable for the first harmonic degrees. Moreover, we can scale the control parameters of the inversion with respect to the degree, in order to make the Moho topography and the density a posteriori verify Kaula type laws. It enables to extrapolate the reliable information upon Moho topography to greater degrees. The method also provides a way to estimate the minimum amount of stress difference needed in order to adjust gravimetric and topographic models.

References

Backus, G. E., 1966, Potentials for tangent tensor fields on spheroids, Arch. Ration. Mech. Analysis, 22, 210-252.

Backus, G. E., 1967, Converting vector and tensor equations to scalar equations in spherical coordinates, Geophys. J. R. Astron. Soc., 13, 71-101.

Chambat F., and Valette B., 2001, Mean radius, mass and inertia for reference Earth's models, Phys. Earth. Planet. Inter., Vol 124/3-4, pp 237-253.

Dahlen, F. A., 1981, Isostasy and the ambient state of stress in the oceanic lithosphere, J. Geophys. Res., 86, B9, 7801-7807.

Dahlen, F. A., 1982, Isostatic geoid anomalies on a sphere, J. Geophys. Res., 87, B5, 3943-3947.

Jeffreys, H., 1976, The Earth, 6th Edition, Cambridge University Press, London.

Poincaré, H., 1902, Figure d'équilibre d'une masse fluide, Reprint, Jacques Gabay, 1990, Paris.