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Mean radius, mass, and inertia for reference Earth models

Frédéric Chambat[a], Bernard Valette[b,c,*]

[a] Laboratoire de Sciences de la Terre, ENS-Lyon, 46 Allée d'Italie, 69364 Lyon Cedex 07, France
[b] LGIT-Savoie, Universit de Savoie, 73376 Le Bourget-du-Lac Cedex, France
[c] I.R.D., France

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Abstract

This paper focuses on the global quantities, radius, mass, and inertia that are needed for the construction of reference Earth density models. We recall how these quantities b, $\mathcal{M}$, and $\mathcal{I}$ are measured and we give realistic estimates and uncertainties. Since a reference model corresponds to a spherical average of the real Earth, we detail how these estimates need to be corrected in order to be used as input data for such a mean model. The main independent data to be used for reference models are: $b = 6\,371\,230 \pm 10\,\mathrm{m}$, $\mathcal{M}_0 = (5.9733 \pm 0.0090) \times 10^{24}\,\mathrm{kg}$, $\mathcal{I}_0/\mathcal{M}_0 = (1.342\,354 \pm 0.000\,031) \times 10^{13}\,\mathrm{m}^2$.

Keywords: Radius; Mass; Inertia; Reference models; Density; Perturbations

1. Introduction

A reference or mean Earth model is a spherically symmetric model of the Earth's physical parameters. Mean models (e.g. PREM, Dziewonski and Anderson, 1981) are fundamental because, as recalled by Khan (1983), they 'serve as a multidisciplinary framework of reference for Earth's primary physical properties and their manifestations'. At the present time, the need for a new reference Earth model is underlined by the REM website (http://earthref.org/). In an introductory paper on reference Earth models Bullen (1974) pointed out that 'a first requirement of a reference Earth model is that it must fit the mean radius, mass and inertia'. Following a suggestion by Khan (1982), who found large discrepancies between such models, it would be desirable to standardize 'the density models by adopting a uniform data set for the radius, mass and inertia'. This was the purpose of Jeffreys (1976) and Romanowicz and Lambeck (1977) and later Khan (1983), who recommended values and associated uncertainties. Furthermore, these mean data can be used independently from any other observation to determine bounds on the density and its moments inside the Earth, as was done by Valette (2000) using a method initiated by Stieltjes (1884).

The refinement of geodetic and astronomic measurements and of seismological models incites to reconsider the estimates for the radius b, the mass $\mathcal{M}$, the inertia $\mathcal{I}$, the inertia coefficient $\mathcal{I}/\mathcal{M}b^2$, and the ratio $\mathcal{I}/\mathcal{M}$. This is the first purpose of the paper.

The second purpose is to define and estimate the equivalent data $\mathcal{M}_0, \mathcal{I}_0, \mathcal{I}_0/\mathcal{M}_0$ corresponding to the reference density model. Such a model corresponds to an Earth's spherical average which must first be defined. Thus, $\mathcal{M}_0$ and $\mathcal{I}_0$ related by definition to the

* Corresponding author.
E-mail addresses: frederic.chambat@ens-lyon.fr (F. Chambat), bvale@univ-savoie.fr (B. Valette).

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spherical model are not exactly equal to $\mathcal{M}$ and $\mathcal{I}$. We estimate upper bounds for these differences by a second-order shape perturbation. This yields values and uncertainties for $\mathcal{M}_0, \mathcal{I}_0$ and $\mathcal{I}_0/\mathcal{M}_0$.

The concept of mean models is defined in Section 2. In Section 3, we give estimates and standard deviations of the mass, inertia, and mean radius of the Earth. Section 4 is devoted to the calculation of first and second-order perturbations between the real Earth and the mean model. In the last sections, numerical estimates of second-order terms are given, taking into account hydrostatic and non-hydrostatic terms. An original presentation of first-order hydrostatic theory is also given.

Throughout the paper, the Earth is considered as the union of its solid and liquid parts. The classical notations used are: G: gravitational constant; $||x||$: Euclidean norm of vector x; r, θ, λ: spherical coordinates (radius, colatitude, longitude); p_l^m: Legendre function of degree l and order m; $p_l^m(\cos\theta)\cos m\lambda$ and $p_l^m(\cos\theta)\sin m\lambda$: real spherical harmonics normalized as

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} p_l^m(\cos\theta)p_{l'}^{m'}(\cos\theta) \times \mathcal{C}(m\lambda)\mathcal{C}'(m'\lambda)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\lambda = \delta_l^{l'}\delta_m^{m'}\delta_{\mathcal{C}}^{\mathcal{C}'}, \qquad (1)$$

where δ_i^j is the Kronecker symbol, and $\mathcal{C}$ and $\mathcal{C}'$ stand for either sine (when $m \neq 0$) or cosine functions. With this normalization, the expressions of degree 0, 1, and 2 Legendre functions are

$$\begin{aligned} &p_0^0(\cos\theta) = 1, \qquad p_1^0(\cos\theta) = \sqrt{3}\cos\theta,\\ &p_1^1(\cos\theta) = \sqrt{3}\sin\theta,\\ &p_2^0(\cos\theta) = \tfrac{\sqrt{5}}{2}(3\cos^2\theta - 1),\\ &p_2^1(\cos\theta) = \sqrt{15}\sin\theta\cos\theta,\\ &p_2^2(\cos\theta) = \tfrac{\sqrt{15}}{2}\sin^2\theta. \end{aligned} \qquad (2)$$

The degree 0 coefficient of a function $h(\theta, \lambda)$ is denoted $h|_0$:

$$h|_0 = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} h(\theta,\lambda)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\lambda. \qquad (3)$$

2. Definition of a mean model

As pointed out by Valette and Lesage (2001), there are several ways of defining a mean model depending on the averaging procedure. They propose the following construction. First, define a continuous set of surfaces S from the center to the real Earth boundary which extrapolates the interfaces. Second, define the mean radius r of S as the spherical mean over the unit sphere of the distance from the center of mass:

$$r = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} ||x(\theta,\lambda)||\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\lambda, \qquad (4)$$

where $x(\theta, \lambda)$ is the position vector of the points of S with respect to the center of mass. Thus, each surface S can be referenced by its radius r, and a point x of S can be referenced by r, θ, λ (see Fig. 1). Let $a(r, \theta, \lambda)$ be the point of the sphere of radius r with the same θ, λ as $x(\theta, \lambda)$. Define the radial vector field $\xi(r, \theta, \lambda)$ as

$$x(r,\theta,\lambda) = a(r,\theta,\lambda) + \xi(r,\theta,\lambda). \qquad (5)$$

The spherical average p_0 of any scalar field p can be defined as the angular average of p over S:

$$p_0(r) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} p(x(r,\theta,\lambda))\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\lambda. \qquad (6)$$

Therefore r and p_0 are the spherical harmonic degree 0 coefficient of $||x||$ and $p(x)$:

$$r = ||x|||_0, \qquad p_0(r) = p(x)|_0. \qquad (7)$$

As shown by Valette and Lesage (2001), this defines the mean model which is unique to the first-order in ξ with respect to the choice of the family of surface S.

Let denote by b the radius of the model, by ρ_0 and ρ the density of the model and of the real Earth, respectively, and by V_0 and V their domain. The masses are, respectively, defined by

$$\mathcal{M}_0 = \int_{V_0}\rho_0\,\mathrm{d}V = 4\pi\int_0^b \rho_0(r)r^2\,\mathrm{d}r, \qquad (8)$$

$$\mathcal{M} = \int_V \rho\,\mathrm{d}V. \qquad (9)$$

The inertia tensor is defined as

$$\mathcal{J} = \int_V \rho(x^2 I - x\otimes x)\,\mathrm{d}V, \qquad (10)$$

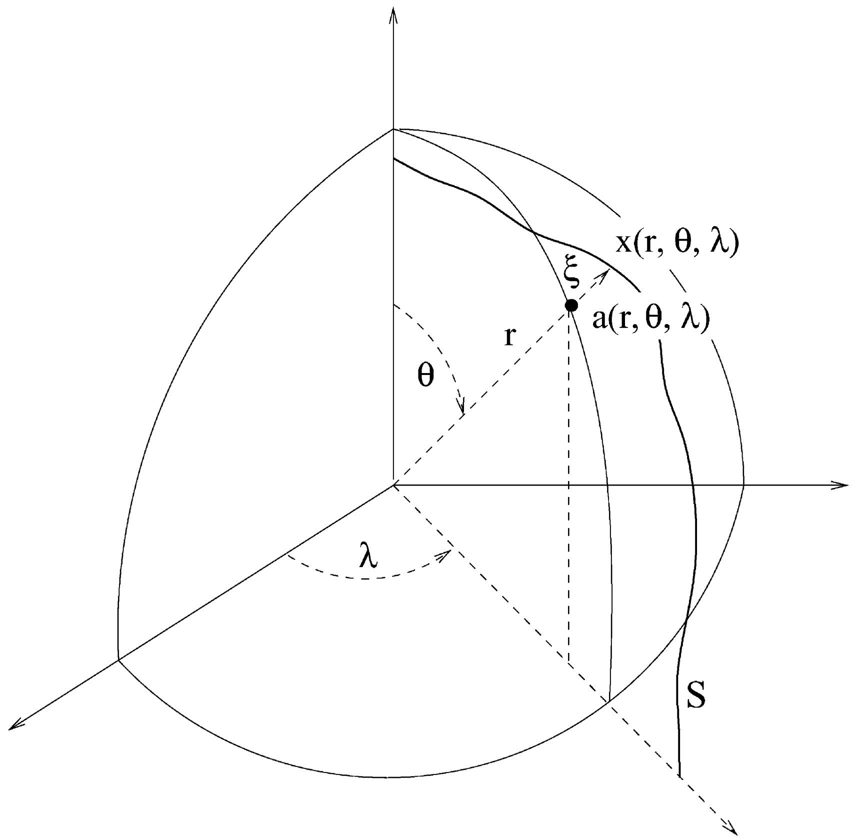

Fig. 1. Notations used to define the reference configuration: the surfaces S which extrapolate the Earth interfaces have mean radii r; the points x of S are referenced by the points a on the spheres of radii r; $\xi = x - a$ is the radial Lagrangian vector between the two configurations; θ is the colatitude and λ the longitude.

where I is the identity tensor and $\otimes$ the tensorial product. Let (e_1, e_2, e_3) be the cartesian basis associated to the equatorial directions $\theta = \pi/2, \lambda = 0$ or $\pi/2$ and to the polar direction $\theta = 0$. Let $A < B < C$ be the inertia moments, i.e. the eigenvalues of the inertia tensor, so that

$$\mathrm{tr}\,\mathcal{J} = \mathcal{J}_{11} + \mathcal{J}_{22} + \mathcal{J}_{33} = A + B + C. \quad (11)$$

The inertia is defined as

$$\mathcal{I} = \frac{1}{3}\,\mathrm{tr}\,\mathcal{J} = \frac{2}{3}\int_V \rho x^2\,\mathrm{d}V, \quad (12)$$

and the mean inertia as

$$\mathcal{I}_0 = \frac{2}{3}\int_{V_0} \rho_0 r^2\,\mathrm{d}V = \frac{8\pi}{3}\int_0^b \rho_0(r) r^4\,\mathrm{d}r. \quad (13)$$

The main purposes of the paper are, firstly the evaluation of $\mathcal{M}$ and $\mathcal{I}$ (Section 3), secondly the evaluation of the differences $\delta\mathcal{M} = \mathcal{M} - \mathcal{M}_0$ and $\delta\mathcal{I} = \mathcal{I} - \mathcal{I}_0$ (Sections 4–6).

3. Radius, mass, and inertia estimates

3.1. Radius b

The mean Earth radius is usually considered to be equal to 6 371 001 m. This value corresponds to the radius of the sphere having the same volume as the reference ellipsoid. As noticed, e.g. by Fan (1998), this ellipsoid is defined in order to best fit the geoid, and does not take the continental lands outside the geoid into account. Thus, the mean radius R of the reference ellipsoid is an estimate of the mean radius of the geoid, and the Earth topography has a mean radius $b > R$. More precisely, denoting by h the altitude and by $h|_0$ its mean value (3),

$$b = R + h|_0, \quad (14)$$

since $h \ll R$. Some authors have already proposed a radius including the continental volume above the geoid. For instance, Marchal (1996) used $b = 6\,371\,200$ m and Fan (1998) determined the ellipsoid which best fits the topography with a mean radius 230 m larger than R.

Numerical estimates of b thus rely on the following:

1. R from the reference ellipsoid parameters;
2. $h|_0$ from global digital elevation models (DEM).

We discuss points 1 and 2 together with the appreciation of their uncertainties. First of all, let us note that the difference between geoid and quasi-geoid is small (e.g. Heiskanen and Moritz, 1967) and can be neglected with respect to the errors on the DEMs.

3.1.1. Mean ellipsoidal radius

A reference ellipsoid is defined by four constants: the equatorial radius a_e (usually designed by a), the geocentric gravitational constant $G\mathcal{M}$, the dynamical form factor J_2, and the angular velocity Ω. The geodetic reference system 1980 (GRS80, e.g. Moritz, 1988) recommends

$$a_e = 6\,378\,137\ \text{m}, \quad (15)$$

$$G\mathcal{M} = 398\,600\,500 \times 10^6\ \text{m}^3\,\text{s}^{-2}, \quad (16)$$

$$J_2 = 1\,082\,630 \times 10^{-9}, \quad (17)$$

$$\Omega = 7\,292\,115 \times 10^{-11}\ \text{rad}\,\text{s}^{-1}. \quad (18)$$

Although, these values are conventional constants, they have been chosen as the most representative of the Earth's parameters. Chovitz (1988) reviewed possible improvements and errors on them, and gave

$$a_e = 6\,378\,136 \pm 1\ \text{m}, \quad (19)$$

$$G\mathcal{M} = (398\,600\,440 \pm 3) \times 10^6\ \text{m}^3\,\text{s}^{-2}, \quad (20)$$

$$J_2 = (1\,082\,626 \pm 2) \times 10^{-9}, \quad (21)$$

$$\Omega = (7\,292\,115 \pm 0.1) \times 10^{-11}\ \text{rad}\,\text{s}^{-1}. \quad (22)$$

Moreover, Chovitz (1988) reported the following value ranges for a_e: $6\,378\,137.8 \pm 2.6$ m, $6\,378\,136.2 \pm (0.5\text{–}1)$ m, $6\,378\,134.8 \pm 2.5$ m, $6\,378\,137.4$ m (unstated uncertainty, from Rapp). Consequently, we adopt the GRS80 value with an uncertainty approximately corresponding to the whole range of reported values:

$$a_e = (6\,378\,137 \pm 3)\ \text{m}, \quad \frac{\mathrm{d}a_e}{a_e} = 4.7 \times 10^{-7}. \quad (23)$$

This uncertainty renders insignificant the 13 cm difference between the mean and the tide-free values (e.g. Groten, 2000).

GRS80 also yields the polar radius a_p (usually denoted b):

$$a_p = 6\,356\,752.3141\ \text{m}, \quad (24)$$

the flattening:

$$f = \frac{a_e - a_p}{a_e} = 0.00335281068118, \quad (25)$$

and the squared second excentricity:

$$e'^2 = \frac{a_e^2 - a_p^2}{a_p^2} = 0.00673949677548. \quad (26)$$

Various mean radii of an ellipsoid can be defined. GRS80 proposes the following ones: the arithmetic mean radius:

$$R_1 = \tfrac{1}{3}(2a_e + a_p) = a_e(1 - \tfrac{1}{3}f) \simeq 6\,371\,008.8\ \text{m}, \quad (27)$$

the equisurfacic sphere radius:

$$R_2 \simeq 6\,371\,007.2\ \text{m}, \quad (28)$$

the equivolumetric sphere radius:

$$R_3 = (a_e^2 a_p)^{1/3} = a(1 - f)^{1/3} \simeq 6\,371\,000.8\ \text{m}. \quad (29)$$

Our definition (4) of mean radius yields through an expansion to the sufficient order in excentricity:

$$R_4 \simeq a_e(1 - \tfrac{1}{6}e'^2 + \tfrac{3}{40}e'^4 - \tfrac{5}{112}e'^6) \simeq 6\,370\,994.4\ \text{m}. \quad (30)$$

We thus propose

$$R = 6\,370\,994.4 \pm 3\ \text{m}, \quad \frac{\mathrm{d}R}{R} = 4.7 \times 10^{-7}. \quad (31)$$

Table 1
Spherical harmonic degree 0 coefficients of the altitude, corresponding to several recent DEMs

Model	Angular resolution	Years of development	$h\|_0$ (m)
FNOC	$10'\times10'$	1960s–1984	237.2
ETOPO5	$5'\times5'$	1980s	233.1
Smoothed ETOPO5	$30'\times30'$		231.4
TerrainBase	$5'\times5'$	until1994	234.3
DTM5	$5'\times5'$	until1995	230.7
JGP95E	$5'\times5'$	1994–1995	231.4

3.1.2. Mean altitude and radius

The calculation of $h|_0$ needs an integration over the sphere of a DEM which is usually given in terms of values of altitude or bathymetry on spherical rectangles. Computation of the degree 0 of several recent models leads to the values of Table 1. Moreover, Fan (1998) found 230.2 m with a smoothed version ($30'\times30'$) of DTM5, and Grafarend and Engels (1992) obtained 233.9 m for model TUG87 with an orthonormal basis related to the ellipsoid. The DEMs are documented in the Catalogue of Digital Elevation Data compiled by Bruce Gittings [1] at the NOAA/NGDC web page [2] and at the EGM96 web page. [3]

For those models in Table 1 which give only the mean topography in each rectangle, we have assumed that the rectangles with negative values entirely belong to the oceanic domain and thus that their altitude is null. This implies, on one hand, an overestimation due to the few continental areas under sea level and, on the other hand, a relatively more important underestimation corresponding to rectangles crossed by a coastline. The FNOC and JGP95E models prevent these systematic effects by explicitly distinguishing between continental altitude and bathymetry. Making use of FNOC, we have estimated the first effect to be of the order of 0.03 m and the second of −1.3 m (−1 m with JGP95E). Furthermore, taking the mean over the sphere or over the ellipsoid yields a difference of less than 1 m. Hence, none of these points can account for the discrepancy between estimates of $h|_0$.

Assuming uncorrelated uncertainties, the standard deviation of $h|_0$ can be approximated by $\sigma_{h|_0} \simeq \sigma/\sqrt{3N}$, where N is the number of $5'\times5'$ rectangles, σ the standard deviation for an individual rectangle, and the factor 3 accounts for the proportion of continents. The uncertainties of $5'\times5'$ models can be considered to be less than $\sigma = 300$ m, which yields $\sigma_{h|_0} = 0.06$ m. The dispersion between models is much greater and is thus a more careful basis to appreciate the uncertainty.

Table 2
Earlier published degree 0 coefficients of the altitude

Mean land elevation (m)	Continental area (%)	Year of publication	Inferred $h\|_0$ (m)
771[a]	30[b]	1921	231
875[a]	29.2[a]	1933	255.5
801[a]	30[b]	PC[c]	240
756[d]	29.1[d]	1967	220
726	30.3[e]	1973	220[e]

[a] Reported in Lee and Kaula (1967).
[b] Unstated, assumed value.
[c] Unstated, private communication in Lee and Kaula (1967).
[d] Lee and Kaula (1967).
[e] Balmino et al. (1973).

Consequently, Table 1 suggests that 233 m is a reasonable estimate of $h|_0$ and that the uncertainty is of several metres. Furthermore, this value is consistent with earlier models of topography that yield deviations < 22 m from 233 m (cf. Table 2). We take

$$h|_0 = 233 \pm 7\,\mathrm{m}, \quad \frac{\mathrm{d}h|_0}{h|_0} = 3\times10^{-2}, \tag{32}$$

and hence finally adopt

$$b = 6\,371\,230 \pm 10\,\mathrm{m}, \quad \frac{\mathrm{d}b}{b} = 1.6\times10^{-6}. \tag{33}$$

3.2. Gravitational potential

The external gravitational potential is usually developed on the basis of spherical harmonics. Outside the Earth, the potential φ is harmonic and reads:

$$\varphi(r,\theta,\lambda) = -\sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{1}{r^{l+1}}(C_{lm}\cos m\lambda + S_{lm}\sin m\lambda)\,p_l^m(\cos\theta), \tag{34}$$

where the coefficients C_{lm} and S_{lm} are expressed as

[1] http://www.geo.ed.ac.uk/home/ded.html.
[2] http://www.ngdc.noaa.gov/seg/topo.
[3] ftp://cddisa.gsfc.nasa.gov/pub/egm96/general_info.

$$\begin{pmatrix} C_{lm} \\ S_{lm} \end{pmatrix} = \frac{G}{2l+1} \int_V \rho(r,\theta,\lambda) r^l \begin{pmatrix} \cos m\lambda \\ \sin m\lambda \end{pmatrix} \times p_l^m(\cos\theta)\, \mathrm{d}V. \tag{35}$$

Taking expression (2) into account, the first coefficients can easily be interpreted. The degree 0 is the geocentric gravitational constant

$$C_{00} = G\mathcal{M} = G \int_V \rho\, \mathrm{d}V. \tag{36}$$

The degree 1 coefficients are linked to the position of the center of mass x_0 by

$$\begin{pmatrix} C_{11} \\ S_{11} \\ C_{10} \end{pmatrix} = \frac{G\mathcal{M}}{\sqrt{3}} x_0. \tag{37}$$

Since the center of the reference frame is usually defined as the center of mass, they are effectively null. The degree 2 coefficients are related to the components of $\mathcal{J}$ in (e_1, e_2, e_3) by the well-known relations (e.g. Heiskanen and Moritz, 1967; Soller, 1984):

$$\mathcal{J}_{11} = \mathcal{I} + \frac{\sqrt{5}}{3G}(C_{20} - \sqrt{3}C_{22}), \tag{38}$$

$$\mathcal{J}_{22} = \mathcal{I} + \frac{\sqrt{5}}{3G}(C_{20} + \sqrt{3}C_{22}), \tag{39}$$

$$\mathcal{J}_{33} = \mathcal{I} - 2\frac{\sqrt{5}}{3G}C_{20}, \qquad \mathcal{J}_{12} = -\sqrt{\frac{5}{3}}\frac{S_{22}}{G}, \tag{40}$$

$$\mathcal{J}_{13} = -\sqrt{\frac{5}{3}}\frac{C_{21}}{G}, \qquad \mathcal{J}_{23} = -\sqrt{\frac{5}{3}}\frac{S_{21}}{G}. \tag{41}$$

Geodesists use dimensionless coefficients defined as

$$\begin{pmatrix} \bar{C}_{lm} \\ \bar{S}_{lm} \end{pmatrix} = \frac{1}{G\mathcal{M}\, a_{\mathrm{e}}^{*l}} \begin{pmatrix} C_{lm} \\ S_{lm} \end{pmatrix}, \tag{42}$$

where a_{e}^* is a reference length usually chosen as the equatorial radius of a reference ellipsoid. However, it is actually $(C_{lm}, S_{lm}) = G\mathcal{M}\, a_{\mathrm{e}}^{*l}(\bar{C}_{lm}, \bar{S}_{lm})$ that are measured and that contain independent information. These values are determined by both satellite tracking data and Earth surface data. Geopotential models are practically given by the values of $\bar{C}_{lm}$ and $\bar{S}_{lm}$ with

$$\varphi(r,\theta,\lambda) = -\frac{G\mathcal{M}}{r}\left\{1 + \sum_{l=2}^{\infty}\sum_{m=0}^{l}\left(\frac{a_{\mathrm{e}}^*}{r}\right)^l (\bar{C}_{lm}\cos m\lambda + \bar{S}_{lm}\sin m\lambda)\, p_l^m(\cos\theta)\right\}. \tag{43}$$

Table 3
Review of some values of $G\mathcal{M}$ (including the atmosphere).

Reference (and name of the gravitational model)	$G\mathcal{M}$ and uncertainty ($10^6\,\mathrm{m}^3\,\mathrm{s}^{-2}$)	
Lerch et al. (1978)	398 600 440	20
Smith et al. (1985)[a]	398 600 434	2
Marsh et al. (1985)[a]	398 600 434	5
Tapley et al. (1985)[a]	398 600 440	2
Newhall et al. (1987)[a]	398 600 443	6
Ries et al. (1989)[a]	398 600 440.5	1
Marsh et al. (1989) (GEM-T2)[b]	398 600 436	
Rapp et al. (1991) (OSU91)[c]	398 600 440	
Ries et al. (1992)[c]	398 600 441.5	0.8
Schwintzer et al. (1997) (GRIM4)[c]	398 600 437.7	0.2
Lemoine et al. (1998) (EGM96)[d]	398 600 443.2	0.4

[a] Reviewed in Ries et al. (1989).
[b] Reported in Rapp and Pavlis (1990).
[c] From the model coefficients file.
[d] This value does not correspond to the one finally adopted for model EGM96 but to the solution found (Lemoine et al., 1998, p. 6–137).

The dynamical form factor J_2 is usually defined as

$$J_2 = -\sqrt{5}\bar{C}_{20} = -\frac{\sqrt{5}}{a_{\mathrm{e}}^{*2}}\frac{C_{20}}{G\mathcal{M}} = \frac{\mathcal{J}_{33} - (1/2)(\mathcal{J}_{11} + \mathcal{J}_{22})}{\mathcal{M}a_{\mathrm{e}}^{*2}}. \tag{44}$$

3.3. Mass $\mathcal{M}$

3.3.1. Determination of $G\mathcal{M}$

Satellite laser ranging tracking data yield the most accurate values of the geocentric gravitational constant (Nerem et al., 1995). A review of some values of $G\mathcal{M}$ (including the atmosphere) is given in Table 3. We retain the current standard value with an uncertainty consistent with the most recent estimates:

$$G\mathcal{M} = (398\,600\,441.5 \pm 4.0) \times 10^6\,\mathrm{m}^3\,\mathrm{s}^{-2}, \quad \frac{\mathrm{d}(G\mathcal{M})}{G\mathcal{M}} = 10^{-8}. \tag{45}$$

3.3.2. Determination of $\mathcal{M}$

The Earth mass is recovered from the value of $G\mathcal{M}$ and of G. Unfortunately, G is a very poorly constrained constant, the uncertainty of which has increased by a factor of about 12 between the 1986 and the 1998 recommended values (Mohr and Taylor, 1999) which yields:

$$G = (6.673 \pm 0.010) \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}},\quad \frac{\mathrm{d}G}{G} = 1.5 \times 10^{-3}. \tag{46}$$

Thus, $\mathcal{M}$ is also poorly constrained:

$$\mathcal{M} = (5.9733 \pm 0.0090) \times 10^{24}\,\mathrm{kg},\quad \frac{\mathrm{d}\mathcal{M}}{\mathcal{M}} = 1.5 \times 10^{-3}. \tag{47}$$

Due to this large uncertainty, the atmospheric mass ($\mathcal{M}_{\mathrm{atm}} = 5.1 \times 10^{18}\,\mathrm{kg} \simeq 8.5 \times 10^{-7}\mathcal{M}$, e.g. Yoder, 1995) is negligible.

3.4. Inertia $\mathcal{I}$

3.4.1. Deviatoric inertia tensor

Relations (38)–(41) include five data (C_{20}, C_{21}, C_{22}, S_{21}, S_{22}) and six unknowns (the components of $\mathcal{J}_{ij}$). The data determine the deviatoric part and the eigendirections of the inertia tensor $\mathcal{J}$, while its trace remains unconstrained. For instance, the eigendirections deduced from geopotential model EGM96 (Lemoine et al., 1998) are as follows:

Longitude $\lambda(^\circ)$	−14.92878	−104.92878	−81.25011
Colatitude $\theta(^\circ)$	90.00003	90.00008	0.00008

Coefficients C_{21}, S_{21} are negligible, so that the eigenvalues of inertia can be expressed as

$$A = \mathcal{I} + \frac{\sqrt{5}}{3G}\left(C_{20} - \sqrt{3}\sqrt{C_{22}^2 + S_{22}^2}\right), \tag{48}$$

$$B = \mathcal{I} + \frac{\sqrt{5}}{3G}\left(C_{20} + \sqrt{3}\sqrt{C_{22}^2 + S_{22}^2}\right), \tag{49}$$

$$C = \mathcal{I} - 2\frac{\sqrt{5}}{3G}C_{20}. \tag{50}$$

3.4.2. The dynamical flattening

The observation of the precession rate enables the determination of the dynamical flattening, defined as

$$\mathcal{H} = \frac{C - (1/2)(A + B)}{C}, \tag{51}$$

and therefore provides an additional information in order to recover the trace of the inertia tensor. The analysis from Dehant and Capitaine (1996) yields:

$$\mathcal{H} = (327\,379 \pm 2) \times 10^{-8},\quad \frac{\mathrm{d}\mathcal{H}}{\mathcal{H}} = 6.1 \times 10^{-6}. \tag{52}$$

As indicated in Section 3.4.1, the quantities $C - (1/2)(A+B)$ and $\mathcal{J}_{33} - (1/2)(\mathcal{J}_{11} + \mathcal{J}_{22})$ can be taken equal to each other (the difference is 3×10^{-12} in relative value). Hence, relation (44) may be rewritten as

$$J_2 = \frac{C - (1/2)(A + B)}{\mathcal{M}a_{\mathrm{e}}^{*2}}, \tag{53}$$

and as

$$C = \mathcal{I} + \tfrac{2}{3}J_2 a_{\mathrm{e}}^{*2}\mathcal{M}. \tag{54}$$

Relations (51) and (53) yield the so-called polar inertia coefficient:

$$\frac{C}{\mathcal{M}a_{\mathrm{e}}^{*2}} = \frac{J_2}{\mathcal{H}}. \tag{55}$$

Taking (54) into account, it also yields the mean inertia coefficient:

$$\frac{\mathcal{I}}{\mathcal{M}b^2} = \frac{J_2}{\mathcal{H}}\left(1 - \frac{2}{3}\mathcal{H}\right)\left(\frac{a_{\mathrm{e}}^*}{b}\right)^2. \tag{56}$$

The term $1 - (2/3)\mathcal{H}$ accounts for the ratio of the polar to the mean moments, while the last term accounts for the ratio of the conventional radius to the real one.

Relation (56) is not based upon a hydrostatic hypothesis and shows that $\mathcal{I}/\mathcal{M}$ is essentially controlled by the ratio of two deviatoric (null in a spherical configuration) data J_2 and $\mathcal{H}$. It also shows that an aspherical model that adjusts the degree 0 and (2, 0) coefficients of the potential fits the mean inertia as long as it fits $\mathcal{H}$. The fact that a hydrostatic model cannot achieve this does not make any difference to the mean model which only depends on $J_2(1 - 2\mathcal{H}/3)/\mathcal{H}$. Therefore, the only question that arises is how to determine the relation between the real Earth mass $\mathcal{M}$ and inertia $\mathcal{I}$ and the data $\mathcal{M}_0$, $\mathcal{I}_0$ relevant to a mean model. This is studied in further sections.

3.4.3. Determination of J_2

In order to compare J_2 from different potential models, we have to define a coefficient J_2^n corresponding to

Table 4
Values of J_2, a_e and related J_2^n from different potential models[a]

Model	a_e^* (m)	Reference year for J_2	In units of 10^{-3}	
			J_2	$J^n_{2(2000)}$
OSU91	6 378 137	1986[b]	1.082 627 04	1.082 626 68
JGM2	6 378 136.3	1986	693	633
GRIM4	6 378 136	1984	719 (22)	643
EGM96	6 378 136.3	1986	668 (8)	608

[a] The models are referenced in Table 2, except for JGM2 (Nerem et al., 1994); the values in parentheses are the uncertainties referred to the last figures of the J_2 value.
[b] Assumed year.

a common value a_e, say the GRS80 $a_e = 6\,378\,137$ m. Moreover, since the secular change in J_2 is estimated to be about $\partial_t J_2 = -2.6 \times 10^{-11}$ per year (Lemoine et al., 1998), and since the J_2 values are given at different times according to the model, a slight correction has to be applied in order to refer them to the same year, say 2000. These corrections, which are of the order of the uncertainty, are expressed as

$$J^n_{2(2000)} = J_{2(2000-\Delta t)} \left(\frac{a_e^*}{a_e}\right)^2 + \partial_t J_2\, \Delta t. \qquad (57)$$

We adopt (see Table 4):

$$J^n_{2(2000)} = (1\,082\,626.4 \pm 0.5) \times 10^{-9}, \quad \frac{\mathrm{d}J_2^n}{J_2^n} = 4.6 \times 10^{-7}. \qquad (58)$$

Another complication is related to the adopted tidal system (e.g. Groten, 2000). Indeed, the following three systems are usually considered:

- the 'mean tide' value J_2^m includes the permanent direct and indirect (due to Earth deformation) luni-solar tides;
- the 'zero-frequency' value J_2^z excludes the direct permanent luni-solar tide potential;
- the 'tide-free' value J_2^n excludes both the direct and indirect permanent luni-solar tidal potential. This is the value given by geopotential models.

If ΔJ_2 represents the direct permanent tides then

$$J_2^m = J_2^z + \Delta J_2, \qquad J_2^z = J_2^n + k_2\, \Delta J_2, \qquad (59)$$

where k_2 is a tidal Love number, conventionally an elastic one, though a fluid one would probably be more convenient. For EGM96 (cf. the 'File description'), the adopted values are $k_2 = 0.3$ and

$$\Delta J_2 = 3.1108 \times 10^{-8}. \qquad (60)$$

The zero-frequency value is the most convenient one for determining the actual inertia (Eq. (56)), since it is related to the actual distribution of mass. The re-introduction of the indirect tide corresponds to a correction of 9×10^{-6} in relative value, which is about 20 times the uncertainty on J_2. It leads to

$$J_2^z = (1\,082\,635.7 \pm 0.5) \times 10^{-9}, \quad \frac{\mathrm{d}J_2^z}{J_2^z} = 4.6 \times 10^{-7}. \qquad (61)$$

3.4.4. Determination of $\mathcal{I}/\mathcal{M}$ and $\mathcal{I}$

The inertia coefficient and the ratio $\mathcal{I}/\mathcal{M}$ are evaluated by Eq. (56) after taking $J_2 = J_2^z$. They are well constrained, since G is not involved:

$$\frac{\mathcal{I}}{\mathcal{M}} = (1.342\,364 \pm 0.000\,009) \times 10^{13}\,\mathrm{m}^2, \quad \frac{\mathrm{d}(\mathcal{I}/\mathcal{M})}{\mathcal{I}/\mathcal{M}} = 6.6 \times 10^{-6}, \qquad (62)$$

$$\frac{\mathcal{I}}{\mathcal{M}b^2} = 0.330\,692 \pm (3 \times 10^{-6}), \quad \frac{\mathrm{d}(\mathcal{I}/\mathcal{M}b^2)}{\mathcal{I}/\mathcal{M}b^2} = 9.7 \times 10^{-6}. \qquad (63)$$

Only $\mathcal{I}/\mathcal{M}$ needs to be slightly corrected for the atmospheric contribution (8.5×10^{-7} in relative value) when considering the liquid–solid Earth. Indeed the ratio corresponding to the Earth without atmosphere can be expressed as a function of the one with atmosphere by

$$\left.\frac{\mathcal{I}}{\mathcal{M}}\right|_{\text{no atm}} \simeq \left.\frac{\mathcal{I}}{\mathcal{M}}\right|_{\text{with atm}} \times \left(1+\frac{\mathcal{M}_{\text{atm}}}{\mathcal{M}}\left(1-\frac{2}{3}\frac{\mathcal{M}b^2}{\mathcal{I}}\right)\right), \tag{64}$$

$$\left.\frac{\mathcal{I}}{\mathcal{M}}\right|_{\text{no atm}} \simeq \left.\frac{\mathcal{I}}{\mathcal{M}}\right|_{\text{with atm}} \left(1-\frac{\mathcal{M}_{\text{atm}}}{\mathcal{M}}\right). \tag{65}$$

This finally leads to

$$\frac{\mathcal{I}}{\mathcal{M}} = (1.342\,363 \pm 0.000\,009) \times 10^{13}\,\text{m}^2. \tag{66}$$

The inertia is determined with much less precision because of the uncertainty on $\mathcal{M}$ (i.e. on G):

$$\mathcal{I} = (8.018 \pm 0.012) \times 10^{37}\,\text{m}^2\,\text{kg}, \quad \frac{\text{d}\mathcal{I}}{\mathcal{I}} = 1.5 \times 10^{-3}. \tag{67}$$

If we suppose that the four data $G\mathcal{M}$, C_{20}, G, and $\mathcal{H}$ are independently determined, then $\mathcal{M}$ and $\mathcal{I}/\mathcal{M}$ are uncorrelated, while the correlation coefficient of $\mathcal{I}$ and $\mathcal{M}$ is approximately

$$1-\frac{1}{2}\left(\frac{\text{d}(\mathcal{I}/\mathcal{M})/(\mathcal{I}/\mathcal{M})}{\text{d}\mathcal{M}/\mathcal{M}}\right)^2, \tag{68}$$

which is very close to 1.

4. Perturbations

4.1. Perturbation formalism

In order to evaluate the significance of the spherical model with respect to the real Earth, we evaluate the differences $\delta\mathcal{M} = \mathcal{M} - \mathcal{M}_0$ and $\delta\mathcal{I} = \mathcal{I} - \mathcal{I}_0$ by using a perturbation approach. For this purpose, we make use of the shape perturbation formalism given by Valette and Lesage (2001). In this approach, the real Earth is related to the reference model by a continuous deformation. Then the physical parameters of the Earth can be derived from those of the reference model through a Taylor expansion. This defines the perturbations to the different orders. The deformation of the Earth domain is parameterized by a scalar t ranging from 0 (for the reference configuration) to 1 (for the real Earth) and which can be thought of as a virtual time. We thus consider the following mapping:

$$\forall (a,t) \in V_0 \times [0,1], \quad (a,t) \to x(a,t) \in V_t, \tag{69}$$

with $\forall a \in V_0$, $x(a,0) = a$, $x(a,1) = x$ (see Fig. 1) and $V_{t=0} = V_0$, $V_{t=1} = V$. For any regular tensorial field T, we can now consider the mapping:

$$\forall (a,t) \in V_0 \times [0,1], \quad (a,t) \to T(x(a,t),t). \tag{70}$$

The Lagrangian displacement of order n is defined as

$$\xi_n(a) = \left.\frac{\text{d}^n}{\text{d}t^n} x(a,t)\right|_{t=0}, \tag{71}$$

and the Eulerian, respectively, Lagrangian, perturbation of order n of T as

$$\delta_{ne} T(a) = \left.\frac{\partial^n}{\partial t^n} T(x(a,t),t)\right|_{t=0}, \tag{72}$$

$$\delta_{n\ell} T(a) = \left.\frac{\text{d}^n}{\text{d}t^n} T(x(a,t),t)\right|_{t=0}. \tag{73}$$

With these notations, it is straightforward to see that

$$\delta_{ne} x = 0 \quad \text{and} \quad \delta_{n\ell} x = \xi_n. \tag{74}$$

Defining ξ, $\delta_e T$, and $\delta_\ell T$, respectively, by

$$x(a,1) = a + \xi(a), \tag{75}$$

$$T(a,1) = T(a,0) + \delta_e T(a), \tag{76}$$

$$T(x(a,1),1) = T(a,0) + \delta_\ell T(a). \tag{77}$$

Taylor expansion yields to the order N:

$$\xi(a) = \sum_{n=1}^{N} \frac{1}{n!} \xi_n(a), \tag{78}$$

$$\delta_e T(a) = \sum_{n=1}^{N} \frac{1}{n!} \delta_{ne} T(a), \tag{79}$$

$$\delta_\ell T(a) = \sum_{n=1}^{N} \frac{1}{n!} \delta_{n\ell} T(a). \tag{80}$$

Consider now a scalar field f and a vectorial field u. It is straightforward to get the following usual relations to the first-order:

$$\delta_{1\ell} f = \delta_{1e} f + \text{grad}\, f \cdot \xi_1, \tag{81}$$

$$\delta_{1\ell}(\operatorname{div} u) = \operatorname{div}(\delta_{1\ell} u) - \operatorname{tr}(\nabla u \cdot \nabla \xi_1). \quad (82)$$

In the case of an integral $\mathcal{F} = \int_V f(x)\,\mathrm{d}V$, one can obtain to the first-order that

$$\delta\mathcal{F} = \delta_1\mathcal{F} = \int_{V_0} (\delta_{1\ell} f + f \operatorname{div} \xi_1)\,\mathrm{d}V, \quad (83)$$

and to the second-order that

$$\delta\mathcal{F} = \delta_1\mathcal{F} + \tfrac{1}{2}\delta_2\mathcal{F}, \quad (84)$$

with

$$\delta_2\mathcal{F} = \int_{V_0} \{\delta_{2\ell} f + 2\delta_{1\ell} f \operatorname{div} \xi_1 + f(\operatorname{div}\xi_2 + (\operatorname{div}\xi_1)^2 - \operatorname{tr}(\nabla\xi_1 \cdot \nabla\xi_1))\}\,\mathrm{d}V. \quad (85)$$

4.2. Perturbation of radius and density

Since the perturbations correspond to a purely mathematical setting, we are free to choose the evolution of the mapping. It is convenient to set $x(a,t) = a + t\xi$ so that the shape perturbation is of first-order only:

$$\xi = \xi_r e_r = \xi_1, \quad \xi_n = 0, \ \forall n \geq 2. \quad (86)$$

In the same way, we assume that

$$\delta_\ell \rho = \delta_{1\ell}\rho, \quad \delta_{n\ell}\rho = 0, \ \forall n \geq 2. \quad (87)$$

Relations (4) and (6) can thus be rewritten as

$$(||x|| - r)|_0 = \xi_r|_0 = 0, \quad (88)$$

$$(\rho(x) - \rho_0(r))|_0 = \delta_\ell\rho|_0 = 0, \quad (89)$$

since $\rho(x) \equiv \rho(x(a,1),1)$ and $\rho_0(r) \equiv \rho(a,0)$.

4.3. Perturbation of mass and inertia

From now on, the subscript 0 corresponding to the reference configuration will be dropped. Using (83) and the relation:

$$\delta_{1\ell}(x^2) = 2x \cdot \delta_{1\ell} x = 2r\xi_r, \quad (90)$$

we obtain

$$\delta_1\mathcal{M} = \int_V (\delta_\ell\rho + \rho \operatorname{div}\xi)\,\mathrm{d}V, \quad (91)$$

$$\delta_1\mathcal{I} = \frac{2}{3}\int_V \left(\delta_\ell\rho + \rho\operatorname{div}\xi + \frac{2\rho\xi_r}{r}\right) r^2\,\mathrm{d}V. \quad (92)$$

It shows, with the help of

$$\operatorname{div}\xi = \partial_r\xi_r + 2\frac{\xi_r}{r}, \quad (93)$$

and (88) and (89) that $\delta_1\mathcal{M} = 0$ and $\delta_1\mathcal{I} = 0$, i.e. the mass and the inertia are preserved to the first-order. Therefore, the total perturbations to the second-order read

$$\delta\mathcal{M} = \delta_1\mathcal{M} + \tfrac{1}{2}\delta_2\mathcal{M} = \tfrac{1}{2}\delta_2\mathcal{M}, \quad (94)$$

$$\delta\mathcal{I} = \delta_1\mathcal{I} + \tfrac{1}{2}\delta_2\mathcal{I} = \tfrac{1}{2}\delta_2\mathcal{I}. \quad (95)$$

On the other hand (85) yields

$$\delta_2\mathcal{M} = \int_V \{2\delta_\ell\rho \operatorname{div}\xi + \rho((\operatorname{div}\xi)^2 - \operatorname{tr}(\nabla\xi\cdot\nabla\xi))\}\,\mathrm{d}V, \quad (96)$$

$$\delta_2\mathcal{I} = \frac{2}{3}\int_V \{\delta_{2\ell}(\rho x^2) + 2\delta_{1\ell}(\rho x^2)\operatorname{div}\xi + \rho((\operatorname{div}\xi)^2 - \operatorname{tr}(\nabla\xi\cdot\nabla\xi))r^2\}\,\mathrm{d}V. \quad (97)$$

By using the relations (90) and (93) and

$$\delta_{2\ell}(x^2) = \delta_{1\ell}(2x\cdot\xi) = 2\xi^2, \quad (98)$$

$$\delta_{2\ell}(\rho x^2) = 2\rho\xi^2 + 4r\xi_r\delta_\ell\rho, \quad (99)$$

$$\operatorname{tr}(\nabla\xi\cdot\nabla\xi) = (\partial_r\xi_r)^2 + 2\left(\frac{\xi_r}{r}\right)^2, \quad (100)$$

we finally deduce that

$$\delta\mathcal{M} = \int_V \left\{\delta_\ell\rho\left(\partial_r\xi_r + 2\frac{\xi_r}{r}\right) + \rho\frac{\xi_r}{r}\left(2\partial_r\xi_r + \frac{\xi_r}{r}\right)\right\}\mathrm{d}V, \quad (101)$$

$$\delta\mathcal{M} = \int_V \left\{\delta_\ell\rho\operatorname{div}\xi + \rho\operatorname{div}\left(\frac{\xi^2}{r}e_r\right)\right\}\mathrm{d}V, \quad (102)$$

$$\delta\mathcal{I} = \frac{2}{3}\int_V \left\{\delta_\ell\rho\left(\partial_r\xi_r + 4\frac{\xi_r}{r}\right) + 2\rho\frac{\xi_r}{r}\left(2\partial_r\xi_r + 3\frac{\xi_r}{r}\right)\right\} r^2\,\mathrm{d}V, \quad (103)$$

$$\delta\mathcal{I} = \frac{2}{3}\int_V \{\delta_\ell\rho\operatorname{div}(\xi r^2) + 2\rho\operatorname{div}(\xi^2 r e_r)\}\,\mathrm{d}V. \quad (104)$$

In order to calculate these shape perturbations, one should ideally know the perturbations in density

$\delta_\ell \rho$ on surfaces of height ξ_r above each reference sphere. We can only evaluate the order of magnitude of these terms. Since the mean model is defined to the first-order only, it will give an estimate of the theoretical error on the data. For this purpose, we separate in the Earth's shape the hydrostatic ellipticity due to the axial rotation from the non-hydrostatic contribution which contains all spherical harmonic degrees.

5. Hydrostatic theory

In this section, the hydrostatic contribution to $\delta\mathcal{M}$ and $\delta\mathcal{I}$ is estimated. Hydrostatic theory consists in solving together Poisson's equation:

$$\Delta\varphi = 4\pi G\rho - 2\Omega^2, \tag{105}$$

and the equilibrium equation:

$$\operatorname{grad} p = -\rho \operatorname{grad}\varphi, \tag{106}$$

with the boundary conditions:

$$[\varphi] = 0, \quad [\operatorname{grad}\varphi \cdot n] = 0, \tag{107}$$

$$[p] = 0, \quad \varphi(x) \sim -\tfrac{1}{2}(\Omega^2 x^2 - (\Omega \cdot x)^2) \text{ at } \infty, \tag{108}$$

where φ is here the gravity potential, p the pressure, Ω the rotation vector, and [ ] denotes the jump across an interface.

5.1. First-order theory

These equations are solved by a first-order perturbation between the non-rotating mean model and the aspherical model rotating with the actual Ω. Let $\delta_e\varphi$ and $\delta_e\rho$ be the degree l, m coefficients of the development of the potential and density Eulerian perturbations. The indices l, m and the dependence in r will usually be dropped. Eqs. (105) and (107) yield for $l \neq 0$:

$$\left\{\partial_r^2 + \frac{2}{r}\partial_r - \frac{l(l+1)}{r^2}\right\}\delta_e\varphi = 4\pi G\delta_e\rho, \tag{109}$$

$$[\delta_e\varphi] = 0, \quad [\operatorname{grad}(\delta_e\varphi) - 4\pi G\rho\xi]\cdot e_r = 0. \tag{110}$$

Let us now consider the new variables:

$$h_\varphi = \frac{\delta_e\varphi}{g}, \qquad \delta_\varphi\rho = \delta_e\rho + h_\varphi\partial_r\rho,$$
$$h = \xi_r - h_\varphi, \tag{111}$$

where g stands for the (negative) radial gravity in the reference state, h_φ the first-order equipotential height above the sphere of radius r, $\delta_\varphi\rho$ the lateral variations of density on the associated equipotential surface, and h the height above the equipotential surface. Thus, for $r = b$, h_φ corresponds to the geoid height and h to the altitude. Using these variables, Eqs. (109) and (110) can be rewritten as

$$\partial_r^2 h_\varphi - \frac{2}{r}(1-3\gamma)\partial_r h_\varphi - \frac{k^2}{r^2}h_\varphi = \frac{4\pi G}{g}\delta_\varphi\rho, \tag{112}$$

$$[h_\varphi] = 0 \text{ et } [\partial_r h_\varphi] = \frac{3[\gamma]h}{r}, \tag{113}$$

$$b\partial_r h_\varphi(b) - 3\gamma h(b) + (l-1)h_\varphi(b) = \frac{\sqrt{5}}{3}\frac{\Omega^2 b^2}{g(b)}\delta_l^2\delta_m^0, \tag{114}$$

$$\begin{pmatrix} h_\varphi \\ r\partial_r h_\varphi \end{pmatrix} \underset{r\to 0}{\overset{\sim \text{cst}}{}} \begin{pmatrix} 1 \\ l-1 \end{pmatrix} r^{l-1}, \tag{115}$$

where

$$k = \sqrt{(l-1)(l+2)} \tag{116}$$

and

$$\gamma = -\frac{4\pi G\rho r}{3g} = \frac{\rho(r)}{\rho_2(r)} \tag{117}$$

is the ratio of the reference density ρ to the mean density $\rho_2(r) = 3\int_0^r \rho s^2\,\mathrm{d}s/r^3$ inside the sphere of radius r. The hydrostatic hypothesis (106) implies that equidensity, equipotential, and equipressure surfaces all coincide with each other and that the interfaces are equiparameter surfaces, so that $\delta_\varphi\rho \equiv 0$ and $h \equiv 0$. Then equations (112)–(115) correspond to Clairaut's equation, the solution of which is given for all $l \neq 0$ by

$$\begin{pmatrix} h_\varphi \\ r\partial_r h_\varphi \end{pmatrix}(r) = \frac{\sqrt{5}}{3}\frac{\Omega^2 b^2}{g(b)}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(r), \tag{118}$$

where x_1, x_2 verify

$$\frac{\mathrm{d}}{\mathrm{d}r}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{r}\begin{pmatrix} 0 & 1 \\ k^2 & 3(1-2\gamma) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{119}$$

with the conditions:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \underset{r \to 0}{\overset{\sim \mathrm{cst}}{}} \begin{pmatrix} 1 \\ l-1 \end{pmatrix} r^{l-1}, \tag{120}$$

$$[x_1] = 0, \quad [x_2] = 0 \ \text{ at the interfaces}, \tag{121}$$

$$(x_2 + (l-1)x_1)(b) = \delta_l^2 \delta_m^0 \ \text{ at } \ r = b. \tag{122}$$

5.2. A Poincarés result

A result stated by Poincaré (1902) shows that the solutions of this system are bounded by simple functions of the radius. Let us assume that for any r, the ratio γ (117) verifies $0 \leq \gamma(r) \leq 1$, i.e. that $0 \leq \rho(r) \leq \rho_2(r)$, or that ρ_2 decreases with r. This hypothesis is clearly weaker than the one of a decreasing density. Let γ_0 be the minimum over $[0, b]$ of $\gamma(r)$ and p be defined as

$$p = \tfrac{1}{2}\left(3(1-2\gamma_0) + \sqrt{9(1-2\gamma_0)^2 + 4k^2}\right). \tag{123}$$

Under the above hypothesis, the solution $(x_1, x_2)(r)$ of the system (119) with the conditions (120) and (121) verifies (Poincaré, 1902,[4] cf. Appendix A):

$$l - 1 \leq \frac{x_2(r)}{x_1(r)} \leq p \leq l + 2. \tag{124}$$

Using $x_2 = r\,\mathrm{d}x_1/\mathrm{d}r$, it yields by integration:

$$\left(\frac{r}{b}\right)^{l+2} \leq \left(\frac{r}{b}\right)^{p} \leq \frac{x_1(r)}{x_1(b)} \leq \left(\frac{r}{b}\right)^{l-1}. \tag{125}$$

It implies that the condition (122) can only be fulfilled for $l = 2$ and $m = 0$ and thus that as is well known (see for instance Jeffreys, 1976), the solution only contains the degree 2 order 0 term. Another immediate consequence is that the internal flattening, defined as

$$\epsilon(r) = -\frac{3\sqrt{5}}{2}\frac{h_\varphi|_2^0}{r} \tag{126}$$

is positive and increases with the radius:

$$\frac{\partial_r \epsilon}{\epsilon} = \frac{\partial_r h_\varphi|_2^0}{h_\varphi|_2^0} - \frac{1}{r} = \frac{(x_2/x_1) - 1}{r} \geq 0. \tag{127}$$

[4] This result is slightly more general than the one given by Poincaré (1902) (p. 84).

5.3. Ellipticity corrections

The mass and inertia perturbations (101) and (103) can be estimated in the hydrostatic case. Taking the surfaces S as the equipotential surfaces, i.e. $\xi_r = h_\varphi p_2^0(\cos\theta)$ and $\delta_\ell \rho = 0$, implies:

$$\delta\mathcal{M} = 4\pi \int_0^b \rho \frac{h_\varphi}{r}\left(2\partial_r h_\varphi + \frac{h_\varphi}{r}\right) r^2\,\mathrm{d}r, \tag{128}$$

$$\delta\mathcal{I} = \frac{16\pi}{3} \int_0^b \rho \frac{h_\varphi}{r}\left(2\partial_r h_\varphi + 3\frac{h_\varphi}{r}\right) r^4\,\mathrm{d}r. \tag{129}$$

One can easily obtain theoretical bounds on these integrals by noting that $\partial_r h_\varphi$ and h_φ are both negative and by using inequalities (124) and (125). This yields:

$$0 < \frac{\delta\mathcal{M}}{\mathcal{M}} \leq \frac{4}{45}(2p+1)\epsilon^2(b), \qquad 0 < \frac{\delta\mathcal{I}}{\mathcal{I}} \leq \frac{8}{45}(2p+3)\epsilon^2(b). \tag{130}$$

Using $\epsilon(b) \simeq 1/300$, the values of these bounds are $\delta\mathcal{M}/\mathcal{M} \leq 5.0 \times 10^{-6}$ and $\delta\mathcal{I}/\mathcal{I} \leq 13.8 \times 10^{-6}$ with $p = 2$ ($\gamma_0 = 1/2$), $\delta\mathcal{M}/\mathcal{M} \leq 3.0 \times 10^{-6}$ and $\delta\mathcal{I}/\mathcal{I} \leq 9.9 \times 10^{-6}$ with $p = 1$ ($\gamma_0 = 1$).

On other hand, a numerical integration of (118)–(122) yields:

$$\frac{\delta\mathcal{M}}{\mathcal{M}} \simeq 2.7 \times 10^{-6}, \qquad \frac{\delta\mathcal{I}}{\mathcal{I}} \simeq 9.4 \times 10^{-6}, \tag{131}$$

$$\frac{\delta(\mathcal{I}/\mathcal{M})}{\mathcal{I}/\mathcal{M}} = \frac{\delta\mathcal{I}}{\mathcal{I}} - \frac{\delta\mathcal{M}}{\mathcal{M}} \simeq 6.7 \times 10^{-6}. \tag{132}$$

These values are very close to the ones for the homogeneous case ($p = 1$), because $h_\varphi(r)$ is numerically close to the homogeneous solution $h_\varphi(r) = h_\varphi(b)r/b$.

6. Estimation of $\mathcal{I}_0/\mathcal{M}_0$

The purpose is now to numerically estimate the perturbations (102) and (104) and to put bounds on them in the general non-hydrostatic framework. Let us first decompose ξ_r as

$$\xi_r = \xi_h + \xi_d, \tag{133}$$

where ξ_h is the degree 2 order 0 component which corresponds approximately to the hydrostatic ellipticity.

ξ_d is related to the deviatoric part of the stress tensor. Integrating by parts the term of (102) containing ξ_d^2 yields:

$$\delta\mathcal{M} = \delta_h\mathcal{M} + \delta_d\mathcal{M}, \tag{134}$$

with

$$\delta_h\mathcal{M} = \int_V \rho\frac{\xi_h}{r}\left(2\partial_r\xi_h + \frac{\xi_h}{r}\right)\mathrm{d}V,$$
$$\delta_d\mathcal{M} = \int_V \delta_\ell\rho\,\mathrm{div}(\xi_h e_r)\,\mathrm{d}V + \int_V \delta_\ell\rho\,\mathrm{div}(\xi_d e_r)\,\mathrm{d}V - \int_V \partial_r\rho\frac{\xi_d^2}{r}\,\mathrm{d}V - \sum_{r_\Sigma}\int_\Sigma [\rho]\frac{\xi_d^2}{r_\Sigma}\,\mathrm{d}\Sigma, \tag{135}$$

where the r_Σ are the radii of the interfaces.

The term $\delta_h\mathcal{M}$, which corresponds to the hydrostatic shape, is positive and has been calculated in Section 5.3.

By using the approximation $\xi_h \simeq h_\varphi(b)(r/b)p_2^0$, we see that the first integral in $\delta_d\mathcal{M}$ involves the product of $3h_\varphi(b)/b$ by the integral of the degree 2 order 0 coefficient of $\delta_\ell\rho$. This last term is a priori oscillating with the radius and is of the order of $\rho\xi_d/r$, i.e. much smaller than $\rho h_\varphi(b)/b$. Hence the whole term is much smaller than $(h_\varphi(b)/b)^2 \simeq 10^{-6}$ in relative value to $\mathcal{M}$. In fact, we choose to take the last integral in $\delta_d\mathcal{M}$ as an upper bound for this term. The second integral is a priori of the order of magnitude of the last two terms. One may note that since the two fields $\delta_\ell\rho$ and $\mathrm{div}(\xi_d e_r)$ are probably not well correlated, compensations can occur in the integral, while the last two integrals contain only negative terms.

These last two integrals both depend on the radial variation of the density and on the spherical quadratic norm of ξ_d. The main term corresponds to the surfacic term, since the radial variation of density mostly occurs at interfaces. Let us note that this term, which corresponds to a piecewise homogeneous Earth, can also be deduced with the simple method given by Balmino (1994). Denoting

$$\delta_\Sigma\mathcal{M} = -\int_\Sigma [\rho]\frac{\xi_d^2}{r_\Sigma}\,\mathrm{d}\Sigma, \tag{136}$$

a reasonable upper bound is thus

$$|\delta_d\mathcal{M}| \le 4\delta_\Sigma\mathcal{M}. \tag{137}$$

It remains to estimate $\delta_\Sigma\mathcal{M}$, which can be rewritten as

$$\delta_\Sigma\mathcal{M} = -4\pi\sum_{r=r_\Sigma} r_\Sigma[\rho](\xi_d^2)|_0. \tag{138}$$

The different values of the RMS $\sqrt{(\xi_d^2)|_0}$ can be practically evaluated from interface models, either by direct integration over the sphere or by summing up the squared spherical harmonics coefficients, according to the way the models are specified. It yields 0.63 km for the external topography and 2.51 km for the solid topography with model JGP95E, 12.2 km for the Moho with CRUST5.1 (Mooney et al., 1998), 4.8–6.0 and 7.2 km for the 410 and 660 km discontinuities with Flanagan and Shearer (1998, 1999) models. We suppose that for the CMB, this value does not exceed 1 km. This set of values leads to

$$\frac{\delta_\Sigma\mathcal{M}}{\mathcal{M}} \le 1.1\times 10^{-6}. \tag{139}$$

In the same way, and with similar notations, we can infer that

$$|\delta_d\mathcal{I}| \le 4\delta_\Sigma\mathcal{I}, \tag{140}$$

with

$$\delta_\Sigma\mathcal{I} = -\frac{16\pi}{3}\sum_{r=r_\Sigma} r_\Sigma^3[\rho](\xi_d^2)|_0. \tag{141}$$

Hence

$$\frac{\delta_\Sigma\mathcal{I}}{\mathcal{I}} \le 4.1\times 10^{-6}. \tag{142}$$

Moreover, using (138) and (141), it is straightforward to show that

$$0 < \frac{\delta_\Sigma\mathcal{I}}{\mathcal{I}} < \frac{4}{3}\frac{\mathcal{M}b^2}{\mathcal{I}}\frac{\delta_\Sigma\mathcal{M}}{\mathcal{M}}, \tag{143}$$

and that the second inequality is numerically not far from an equality. This suggests that in $\delta_d(\mathcal{I}/\mathcal{M})$, there is a compensation between perturbations corresponding to the last two positive terms of (135) in such a way that

$$\left|\frac{\delta_d(\mathcal{I}/\mathcal{M})}{\mathcal{I}/\mathcal{M}}\right| \le 2\left(\left|\frac{\delta_\Sigma\mathcal{I}}{\mathcal{I}} - \frac{\delta_\Sigma\mathcal{M}}{\mathcal{M}}\right| + \frac{\delta_\Sigma\mathcal{I}}{\mathcal{I}} + \frac{\delta_\Sigma\mathcal{M}}{\mathcal{M}}\right) = 4\max\left(\frac{\delta_\Sigma\mathcal{I}}{\mathcal{I}}, \frac{\delta_\Sigma\mathcal{M}}{\mathcal{M}}\right) \le 1.64\times 10^{-5}. \tag{144}$$

Since the hydrostatic elliptic term is dominant in the Earth's shape and can be precisely determined relatively independently from the mean model, we think that it is worthwhile to correct $\mathcal{I}/\mathcal{M}$ from the corresponding second-order term (132). The non-hydrostatic terms correspond to a theoretical error due to the looseness of the mean model concept and must be added to the observational error. It leads to

$$\frac{\mathrm{d}(\mathcal{I}_0/\mathcal{M}_0)}{\mathcal{I}_0/\mathcal{M}_0} = \frac{\mathrm{d}(\mathcal{I}/\mathcal{M})}{\mathcal{I}/\mathcal{M}} + \left|\frac{\delta_d(\mathcal{I}/\mathcal{M})}{\mathcal{I}/\mathcal{M}}\right| \leq 2.3 \times 10^{-5}, \qquad (145)$$

$$\frac{\mathcal{I}_0}{\mathcal{M}_0} = \frac{\mathcal{I}}{\mathcal{M}}\left(1 - \frac{\delta_h(\mathcal{I}/\mathcal{M})}{\mathcal{I}/\mathcal{M}}\right) = (1.342\,354 \pm 0.000\,031) \times 10^{13}\,\mathrm{m}^2. \qquad (146)$$

The inertia coefficient $\mathcal{I}_0/\mathcal{M}_0 b^2$ can easily be deduced from these values (see Table 6). Note that due to the large uncertainties on $\mathcal{M}$ and $\mathcal{I}$, only the ratios need to be corrected for second-order terms.

7. Secular variations

With the development of geodetic observations, it is becoming usual to estimate the temporal variations of global data. Let us thus consider the secular evolution of the parameters related to mean models.

As far as we know, no temporal variation of $G\mathcal{M}$ or b has been observed yet. Variation of J_2 has already been discussed in Section 3.4.3, while those of G and $\mathcal{H}$ are yet under the observational uncertainties (Chovitz, 1988; Dehant and Capitaine, 1996). These change rates are related to those of mass and inertia by

$$\frac{\partial_t \mathcal{M}}{\mathcal{M}} = \frac{\partial_t (G\mathcal{M})}{G\mathcal{M}} - \frac{\partial_t G}{G}, \qquad (147)$$

$$\frac{\partial_t(\mathcal{I}/\mathcal{M})}{\mathcal{I}/\mathcal{M}} = \frac{\partial_t \mathcal{I}}{\mathcal{I}} - \frac{\partial_t \mathcal{M}}{\mathcal{M}} = \frac{\partial_t J_2}{J_2} - \frac{\partial_t \mathcal{H}}{\mathcal{H}} \frac{1}{1-(2/3)\mathcal{H}}. \qquad (148)$$

Since the order of magnitude of the meteoritic flux is $10^{-17}\mathcal{M}$ per year, the Earth's mass variation rate is negligible. Therefore, $\mathcal{M}$ can be considered as constant ($\partial_t \mathcal{M} = 0$) and the variations of $\mathcal{I}$ are due to the mass redistribution. Let v be the velocity field inside the Earth, then

$$\partial_t \mathcal{I} = \frac{2}{3}\int_V \rho \frac{\mathrm{d}x^2}{\mathrm{d}t}\,\mathrm{d}V = \frac{4}{3}\int_V v \cdot x\,\mathrm{d}m, \qquad (149)$$

and thus

$$\frac{\partial_t(\mathcal{I}/\mathcal{M})}{\mathcal{I}/\mathcal{M}} = \frac{\partial_t \mathcal{I}}{\mathcal{I}} = \frac{4}{3b}\frac{\mathcal{M}b^2}{\mathcal{I}}\frac{1}{\mathcal{M}b}\int_V v \cdot x\,\mathrm{d}m. \qquad (150)$$

Taking $|(1/\mathcal{M}b)\int_V v \cdot x\,\mathrm{d}m| \leq 1\,\mathrm{cm}$ per year gives $|\partial_t \mathcal{I}/\mathcal{I}| \leq 6.3 \times 10^{-9}$ per year, which is about four times less than $\partial_t J_2/J_2$ and very small with respect to its own uncertainty. Thus, we expect $\partial_t \mathcal{H}/\mathcal{H}$ to be negative and of the order of $\partial_t J_2/J_2$.

We can conclude that over a few decades, the variation of the parameters is small with respect to their uncertainties, except for J_2 that needs a slight correction.

8. Conclusion

The observed Earth's mass and inertia have the same relative uncertainty (1.5×10^{-3}) as the gravitational constant. The inertia ratio $\mathcal{I}/\mathcal{M}$, determined by the zero-tide gravity coefficient J_2^z and the precession constant $\mathcal{H}$, is known with more accuracy (6.6×10^{-6}).

A mean Earth model has been defined as the spherical Lagrangian mean of the real Earth. The corresponding mean radius is $b = 6\,371\,230 \pm 10\,\mathrm{m}$. We estimate the data $\mathcal{I}_0$, $\mathcal{M}_0$, $\mathcal{I}_0/\mathcal{M}_0$ associated with this spherical average by a mathematical second-order shape perturbation. Due to the large uncertainty on G, only $\mathcal{I}_0/\mathcal{M}_0$ needs to be corrected with respect to $\mathcal{I}/\mathcal{M}$; it is corrected from hydrostatic ellipticity (6.7×10^{-6} in relative value), while non-hydrostatic terms are added to the observational error (up to 2.3×10^{-5}). $\mathcal{M}_0$ and $\mathcal{I}_0/\mathcal{M}_0$ are independent data, while $\mathcal{I}_0$ and $\mathcal{M}_0$ are strongly correlated. The likely improvement on the accuracy of the measure of G would directly affect the accuracy of $\mathcal{M}_0$. However, theoretical errors on $\mathcal{M}_0$ will probably remain negligible for the near and not-so-near future.

In Table 5, the observational errors, the hydrostatic and the non-hydrostatic perturbations are compared. It shows that the aspherical perturbations are significant for the ratios $\mathcal{I}/\mathcal{M}$ and $\mathcal{I}/\mathcal{M}b^2$ and that their theoretical errors are slightly greater than the observational errors.

Table 5
Corresponding to the parameters of the first line, the columns contain respectively from top to bottom, in relative values, the final uncertainty (observational + theoretical error), the observational error, the upper bound of the second-order term related to non-hydrostaticity, the hydrostatic ellipticity correction, and the influence of a 230 m change in the radius

Data	b	$\mathcal{M}$	$\mathcal{I}$	$\mathcal{I}/\mathcal{M}b^2$	$\mathcal{I}/\mathcal{M}$
Final uncertainty	1.6×10^{-6}	1.5×10^{-3}	1.5×10^{-3}	2.6×10^{-5}	2.3×10^{-5}
Measurement error	1.6×10^{-6}	1.5×10^{-3}	1.5×10^{-3}	9.7×10^{-6}	6.6×10^{-6}
Non-hydrostatic error	0	4.4×10^{-6}	16.4×10^{-6}	16.4×10^{-6}	16.4×10^{-6}
Ellipticity correction	0	2.7×10^{-6}	9.4×10^{-6}	6.7×10^{-6}	6.7×10^{-6}
Influence of $\delta b = 230$ m	3.6×10^{-5}	0	0	7.2×10^{-5}	0

Table 6
Summary of the data for the real Earth and for the reference Earth model[a]

Data	Symbol	Value (uncertainty)	Unit	Relative uncertainty
Real Earth				
Equatorial radius	a_e	6.378 137 (3)	10^6m	4.7×10^{-7}
Geocentric gravitational constant[b]	$G\mathcal{M}$	3.986 004 415 (40)	10^{14} m^3 s^{-2}	1.0×10^{-8}
Gravitational constant	G	6.673 (10)	10^{-11} m^3 kg^{-1} s^{-2}	1.5×10^{-3}
Mass	$\mathcal{M}$	5.9 733 (90)	10^{24} kg	1.5×10^{-3}
Angular velocity	Ω	7.2921150 (1)	10^{-5} rad s^{-1}	1.4×10^{-8}
Tide-free dynamic form factor[c]	J_2^n	1.0 826 264 (5)	10^{-3}	4.6×10^{-7}
Zero-frequency dynamic form factor[c]	J_2^z	1.0 826 357 (5)	10^{-3}	4.6×10^{-7}
Precession constant	$\mathcal{H}$	3.27 379 (2)	10^{-3}	6.1×10^{-6}
Polar inertia coefficient[b]	$C/\mathcal{M}a_e^2$	0.330 698 (2)		6.6×10^{-6}
Second equatorial inertia coefficient[b]	$B/\mathcal{M}a_e^2$	0.329 619 (2)		6.6×10^{-6}
First equatorial inertia coefficient[b]	$A/\mathcal{M}a_e^2$	0.329 612 (2)		6.6×10^{-6}
Inertia coefficient[d]	$\mathcal{I}/\mathcal{M}a_e^2$	0.329 976 (2)		6.6×10^{-6}
Inertia coefficient[d]	$\mathcal{I}/\mathcal{M}b^2$	0.330 692 (3)		9.7×10^{-6}
Mean inertia ratio[d]	$\mathcal{I}/\mathcal{M}$	1.342 363 (9)	10^{13} m^2	6.6×10^{-6}
Mean inertia	$\mathcal{I}$	8.018 (12)	10^{37} m^2 kg	1.5×10^{-3}
Reference earth model				
Mean solid topography	$h\|_0$	233 (7)	10^2m	3.0×10^{-2}
Mean geoidal radius	R	6 370 994.4 (3.0)	10^6m	4.7×10^{-7}
Physical radius	b	6 371 230 (10)	10^6m	1.6×10^{-6}
Mass	$\mathcal{M}_0$	5.9 733 (90)	10^{24} kg	1.5×10^{-3}
Inertia	$\mathcal{I}_0$	8.018 (12)	10^{37} m^2 kg	1.5×10^{-3}
Inertia ratio[d]	$\mathcal{I}_0/\mathcal{M}_0$	1.342 354 (31)	10^{13} m^2	2.3×10^{-5}
Inertia coefficient[d]	$\mathcal{I}_0/\mathcal{M}_0 b^2$	0.330 690 (9)		2.6×10^{-5}
Inertia coefficient[d]	$\mathcal{I}_0/\mathcal{M}_0 R^2$	0.330 714 (8)		2.4×10^{-5}
Second radial density moment[e]	ρ_2	5 514 (8)	10^3 kg m^{-3}	1.5×10^{-3}
Fourth radial density moment[f]	ρ_4	4 558 (7)	10^3 kg m^{-3}	1.5×10^{-3}

[a] The values in parentheses are the uncertainties referred to the last figures of the nominal values.
[b] With atmosphere.
[c] Related to values of $G\mathcal{M}$ and a_e of this table and referred to year 2000.
[d] Without atmosphere.
[e] $\rho_2 = 3\mathcal{M}_0/4\pi b^3$.
[f] $\rho_4 = 15\mathcal{I}_0/8\pi b^5$.

Table 6 summarizes all the values and uncertainties given in the text. The values for $\mathcal{I}/\mathcal{M}$ and $\mathcal{I}_0/\mathcal{M}_0$ differ from those given by Romanowicz and Lambeck (1977), and Khan (1983) by several standard deviations.

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Appendix A

For $l = 1$, it is straightforward to show that $x_1 =$ cst and $x_2 = 0$. In order to prove (124) for $l \geq 2$, let us define

$$x(r) = \frac{x_2(r)}{x_1(r)}. \quad \text{(A.1)}$$

The definition is a posteriori justified by the fact that x remains finite, i.e. x_1 does not vanish. The x verifies the differential equation:

$$\frac{\mathrm{d}x}{\mathrm{d}r} = \frac{1}{r}(k^2 + 3(1-2\gamma)x - x^2), \quad \text{(A.2)}$$

which can be reformulated as

$$\frac{\mathrm{d}x}{\mathrm{d}r} = -\frac{1}{r}(x - x_+)(x - x_-), \quad \text{(A.3)}$$

with

$$x_\pm = \tfrac{1}{2}\left(3(1-2\gamma) \pm \sqrt{9(1-2\gamma)^2 + 4k^2}\right). \quad \text{(A.4)}$$

It yields:

$$\frac{\partial x_+}{\partial \gamma} = \frac{-6x_+}{\sqrt{9(1-2\gamma)^2 + 4k^2}}. \quad \text{(A.5)}$$

Since $x_+ \geq 0$, x_+ is a decreasing function of γ, and thus for any r:

$$0 < x_+(\gamma = 1) = l - 1 \leq x_+(r) \leq x_+(\gamma_0) = p \leq x_+(\gamma = 0) = l + 2. \quad \text{(A.6)}$$

The relation $x_- x_+ = -(l-1)(l+2)$ shows that x_- is a negative increasing function of γ and that

$$-(l+2) \leq x_-(r) \leq -(l-1) < 0. \quad \text{(A.7)}$$

At the center, $\gamma(r = 0) = 1$, so that $x_+ = l-1$, $x_- = -(l+2)$ and by virtue of condition (120), $x(0) = l - 1$. Noting that $\mathrm{d}x/\mathrm{d}r \geq 0$ for $x \in [x_-, x_+]$ and $\mathrm{d}x/\mathrm{d}r < 0$ outside, we finally conclude that $x(r)$ remains in the interval $[l - 1, p]$.

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