

## RESEARCH NOTE

## Empirical 3-D basis for the internal density of a planet

Frédéric Chambat and Yanick Ricard

Laboratoire de Sciences de la Terre, École Normale Supérieure de Lyon, 46 allée d'Italie, F-69364 Lyon, Cedex 07, France.  
E-mails: fchambat@ens-lyon.fr (FC); ricard@ens-lyon.fr (YR)

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## SUMMARY

Various papers have discussed the forward relationships between internal density anomalies of a planet and its external gravity field. The inverse modelling, i.e. finding the internal density anomalies from the external potential is known to be highly non-unique. In this research note, we explain how a 3-D basis can be built to represent the internal density variations that includes a subset that explicitly spans the kernel of the forward gravity operator. This representation clarifies the origin of the non-uniqueness of the gravity sources and implies the existence of a natural minimal norm inverse for the internal density. We illustrate these ideas by comparing a tomographic model of the mantle to the minimal norm density.

**Key words:** density, geoid, gravity, seismic tomography.

## 1 INTRODUCTION

A problem in geodynamics is to relate the internal density of a planet to its external gravity field. Classically, the internal density  $\rho(r, \theta, \phi)$  and the external gravity potential  $V(r, \theta, \phi)$  are both expanded in terms of orthogonal functions on the sphere

$$\rho(r, \theta, \phi) = \sum_{l,m} \rho_{lm}(r) Y_{lm}(\theta, \phi), \quad (1)$$

$$V(r, \theta, \phi) = \sum_{lm} V_{lm} \left( \frac{a}{r} \right)^{l+1} Y_{lm}(\theta, \phi), \quad (2)$$

where  $Y_{lm}(\theta, \phi)$  are the spherical harmonic functions of colatitude  $\theta$  and longitude  $\phi$ ,  $a$  is the planetary radius,  $\rho_{lm}$  are the spherical components of the density at radius  $r$  and  $V_{lm}$  the gravity potential coefficients.

We can deduce the coefficients  $V_{lm}$  as a function of the density spectral components using

$$V_{lm} = \frac{4\pi G}{a} \int_0^a G_l(r) \rho_{lm}(r) r^2 dr, \quad (3)$$

where  $G_l(r)$  are Green's functions. These Green's functions are simply

$$G_l(r) \equiv G_l^S(r) = \frac{1}{2l+1} \left( \frac{r}{a} \right)^l, \quad (4)$$

if we assume that the planet is spherical (hence the superscript 'S'). This Green's function implies that shallow mass anomalies are the most efficient to generate gravity potential anomalies (usually represented as geoid undulations).

Of course, a planet is not spherical. To take into account the surface undulation, the density anomalies can be usefully split into in-

ternal densities  $\rho^i$  and a surface mass  $\sigma(\theta, \phi)$ . This surface mass with spectral components  $\sigma_{lm}$  is the product of the topographic height with the density at the surface of the Earth. Then, at first order, the external potential verifies

$$V_{lm} = \frac{4\pi G}{a} \int_0^a G_l^S(r) \rho_{lm}^i(r) r^2 dr + \frac{4\pi G a}{2l+1} \sigma_{lm}. \quad (5)$$

Internal interfaces can be taken into account in a similar way by introducing equivalent surface mass anomalies, products of the density jumps at these interfaces by their topographies.

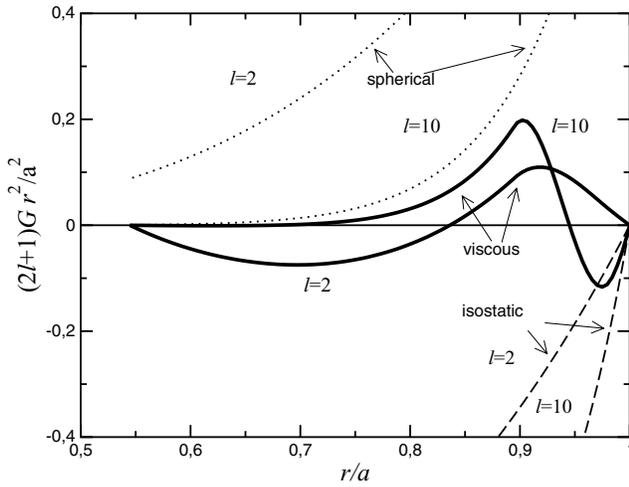
At long wavelength, the planets are close to isostatic equilibrium at least for shallow internal mass anomalies. On the Earth, this equilibrium is reached after a time constant of a few thousand years estimated from modelling the Pleistocene glacial unloading (Cathless 1975). A simple view of isostasy (see Dahlen 1982, for a detailed discussion of the concept of isostasy) indicates that

$$\sigma_{lm} = - \int_{a_i}^a \rho_{lm}^i(r) dr, \quad (6)$$

where  $a_i$  is a radius below which deviatoric stresses are negligible. By replacing this surface load into eq. (5), we see that we can formally still use eq. (3) where only 'true' internal density is considered (i.e.  $\rho$  in eq. 3 is now intended as  $\rho^i$ ) if we use the new Green's function  $G_l^I(r)$  where

$$G_l^I(r) = G_l^S(r) - G_l^S(a) \frac{a^2}{r^2} \quad \text{for } r \geq a_i. \quad (7)$$

This indicates that shallow mass anomalies ( $r \simeq a$ ) do not generate geoid undulations. It also predicts that a dense anomaly at depth should be associated with a geoid low. This is in total opposition to the findings obtained for a spherical non-compensated planet (i.e. when eq. 6 is not verified but one imposes  $\sigma_{lm} = 0$ ).



**Figure 1.** Viscous (thick), spherical (dotted) and isostatic (dashed) Green's functions for degrees 2 and 10. The viscous Green's function are computed for a model where the lithosphere, the upper mantle, and lower mantle viscosities are proportional to 10, 1 and 30.

Isostasy is a very good approximation for shallow masses but, of course, compensation departs from isostasy for deep seated masses, which means that eq. (6) must be replaced by a more general rule. Various papers (Ricard *et al.* 1984; Richards & Hager 1984) have shown that a generalization of the isostatic rule that takes into account sphericity, self-gravitation and the presence of internal interfaces [i.e. the Moho, the core–mantle boundary (CMB). . .] can be obtained for a viscous planet with radial rheological properties. In this theory, the topographies involved in the expression of the potential (as in eq. 5) are computed from the internal density by solving for the mantle flow using a Newtonian viscous law with continuities of the velocity, gravity and stress vector at the interfaces. This enables to write  $V_{lm}$  in the same way as in eq. (3) with a new Green's function  $G_l^\eta(r)$  where the superscript reminds that this dynamic compensation model involves the viscosity profile of the mantle  $\eta$ . Using this approach, a good fit to the observed geoid can be obtained assuming that the internal density of the Earth is related to the observed 3-D seismic velocity structure of the mantle (Forte & Peltier 1987; Hager & Clayton 1989; Ricard *et al.* 1993). Moreover, the geoid, at least at the lowest degrees 2 and 3, is not explained by mass anomalies in the upper mantle because these are compensated by topography anomalies.

Fig. 1 depicts the viscous, spherical and isostatic Green's functions for degrees 2 and 10. The viscous Green's function  $G_l^\eta(r)$  shares with the isostatic Green's function  $G_l^i(r)$  the fact that density anomalies infinitely close to the surface do not generate geoid undulations. Viscous Green's functions also vanish at the CMB as mass anomalies near the core are locally compensated by CMB undulations just like shallow mass anomalies are compensated by surface topography. The viscous Green's functions have much smaller amplitudes than both spherical and isostatic Green's functions. For realistic mantle viscosity profiles, the viscous Green's function at  $l = 2$  is negative in the lower mantle and positive in the upper mantle. At  $l = 10$ , the viscous Green's function looks like the isostatic Green's function for shallow masses, and changes sign with depth where it becomes somewhat similar to the rigid Green's functions.

More general attempts have used 3-D viscosity structures (Zhang & Christensen 1993) or even have tried to remove the necessity of prescribing a rheology (Valette & Chambat 2004). However, the

most successful explanations of the geoid of the Earth are using the simplest model where the rheological properties do not vary laterally.

## 2 THE INVERSE PROBLEM

It is well known that the inverse gravitational problem, i.e. finding the density structure  $\rho(r, \theta, \phi)$  from the gravity coefficients  $V_{lm}$  is an ill-posed problem. This means that there is an infinite number of density distributions inside a planet that can produce a given external gravity field. In front of an ill-posed problem, two general philosophies can be used. On the one hand, one can make profit of some *a priori* knowledge of the density to stabilize the inversion. This can be done for example by assuming that the density remains closely correlated with seismic tomography (this has been used in various papers; see also Ricard & Wuming 1991). On the other hand, one can describe precisely what the null space and what the range of the forward problem are in a Lanczos-type method (Lanczos 1961). This was done for a purely spherical planet, i.e. inverting eq. (3) assuming eq. (4) (Dufour 1977), but to our knowledge has never been discussed when the more appropriate  $G^\eta(r)$  Green's function is used.

This Lanczos-type approach is to build a 3-D basis  $D_{klm}(r, \theta, \phi)$  to represent the internal structure of a planet (the density, but also the seismic  $P$  and  $S$  velocities. . .) in such a way that a subset of this basis is also a basis of the null space of the forward gravity operator.

Let us choose  $D_{klm}(r, \theta, \phi) = R_l^k(r)Y_{lm}(\theta, \phi)$ , then the orthonormalization of this basis means that

$$\int D_{klm}(r, \theta, \phi) D_{k'l'm'}(r, \theta, \phi) dV = 4\pi a^3 \delta_{kk'} \delta_{ll'} \delta_{mm'}, \quad (8)$$

the integral being performed over all the planetary mantle volume (the  $a^3$  term of the right-hand side insures that the  $D_{klm}$  are dimensionless; the  $\delta$  is the Kronecker symbol). The spherical harmonics being already orthogonal on the surface of a sphere, we only need to find  $R_l^k(r)$  functions so that

$$\int_0^a R_l^k(r) R_l^{k'}(r) r^2 dr = a^3 \delta_{kk'}. \quad (9)$$

Of course, we can easily take advantage of our understanding of the gravity field of the Earth by choosing, as one of the  $R_l^k(r)$  functions, the Green's function of the gravitational inverse problem  $G_l^\eta(r)$ . Let us define the norm

$$\|R_l^k\|^2 = \frac{1}{a^3} \int_0^a R_l^k(r) R_l^k(r) r^2 dr \quad (10)$$

and choose

$$R_l^0(r) \equiv \frac{G_l^\eta(r)}{\|G_l^\eta\|}. \quad (11)$$

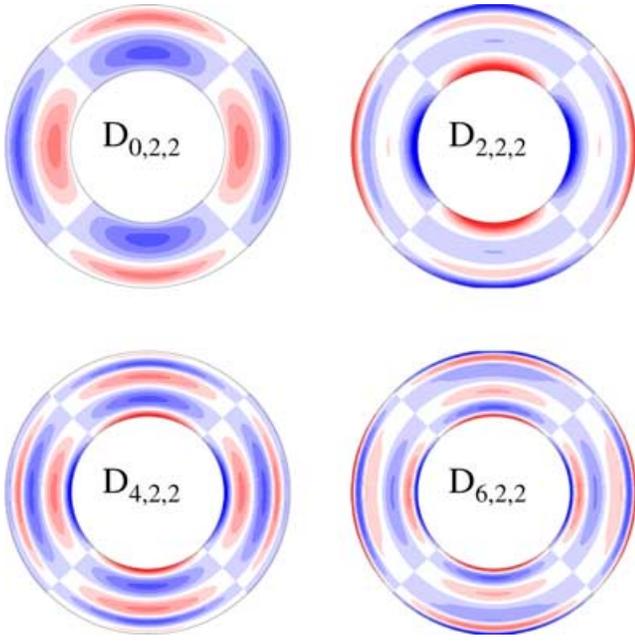
Then, the other  $R_l^k(r)$  can be constructed by a standard orthonormalization procedure starting with linearly independent functions. An easy way is to use Chebychev polynomials  $T_k(r)$  and define  $R_l^k(r)$  as a linear combination of  $T_{k-1}(r)$  and of the previous  $R_l^{k'}(r)$  ( $0 < k' < k$ ),

$$R_l^k(r) = \alpha T_{k-1}(r) + \sum_{0 \leq k' < k} \beta_{k'} R_l^{k'}(r), \quad (12)$$

and deduce the constants  $\alpha$  and  $\beta_{k'}$  from the equations (orthogonality and normalization)

$$\int_0^a R_l^k(r) R_l^{k'}(r) r^2 dr = 0 \quad \text{if } 0 \leq k' < k, \quad (13)$$

$$\|R_l^k(r)\| = 1. \quad (14)$$



**Figure 2.** Equatorial cross-sections of empirical basis functions for  $l = 2$  and  $m = 2$ . Only the first one ( $D_{0,2,2}$ , top left) induces an external gravity field. This first function is based on a gravity Green function computed for a model where the lithosphere, the upper mantle, and lower mantle viscosities are proportional to 10, 1 and 30. The other empirical basis functions (e.g.,  $D_{2,2,2}$ ,  $D_{4,2,2}$ ,  $D_{6,2,2}$ ) span the kernel of the gravity operator; they do not induce any external gravity field.

Fig. 2 depicts the equatorial cross-sections of a few empirical functions for  $l = 2$  and  $m = 2$ . The geoid viscous Green's function is computed following Hager & Clayton (1989) and using a typical mantle viscosity profile (a lithosphere of thickness 100 km with normalized viscosity 10, an upper mantle of reference viscosity 1 and a lower mantle of normalized viscosity 30). The first empirical function  $D_{0,2,2}$  is proportional to this geoid Green's function. By construction  $D_{0,2,2}$  is the only function that induces an  $l = 2$  and  $m = 2$  geoid. The other  $D_{k,l,m}$  (Fig. 2 depicts some of them with  $k = 2, 4$  and  $6$ ) do not induce any gravity signal outside the planet.

By construction, we can replace eq. (1) by

$$\rho(r, \theta, \phi) = \sum_{k,l,m} \rho_{klm} D_{klm}(r, \theta, \phi), \quad (15)$$

where  $\rho_{klm}$  are the components of the density on the empirical basis. This expansion introduced in eq. (3) leads to

$$V_{lm} = 4\pi G a^2 \|G_l^\eta\| \rho_{0lm} \quad (16)$$

and the solution of minimal norm for the inverse problem is simply

$$\rho^{\min}(r, \theta, \phi) = \frac{1}{4\pi G a^2} \sum_{lm} \frac{1}{\|G_l^\eta\|} V_{lm} D_{0lm}(r, \theta, \phi). \quad (17)$$

This minimal norm solution is of course independent of any radial basis and only depends upon the Green's functions. We can define the radial resolution kernel  $R_l(r, r')$  associated with this estimator by

$$\rho_{lm}^{\min}(r) = \int_0^a R_l(r, r') \rho_{lm}(r') dr'. \quad (18)$$

By combining the definitions of the minimal density (eq. 17) with that of the potential (eq. 5), we obtain

$$R_l(r, r') = \frac{1}{a^3} R_l^0(r) R_l^0(r') r'^2 = \frac{1}{\|G_l^\eta\|^2 a^3} G_l^\eta(r) G_l^\eta(r') r'^2. \quad (19)$$

As expected, this resolution does not depend on the radial basis. At a given  $r$ , this resolution is proportional to  $G_l^\eta(r') r'^2$  already plotted in Fig. 1. A very good resolution would imply that  $\rho_{lm}^{\min}$  remains close to the actual density  $\rho_{lm}$  and therefore that  $R_l(r, r')$  is similar to the Dirac function  $\delta(r - r')$ . This is far from being the case as the resolution of the gravity data is indeed very low.

All the solutions of the inverse gravimetric problem can be obtained by adding any component of the null space that has the form

$$\rho^{\text{null}}(r, \theta, \phi) = \sum_{k \neq 0, l, m} n_{klm} D_{klm}(r, \theta, \phi), \quad (20)$$

for any coefficients  $n_{klm}$ .

### 3 APPLICATION

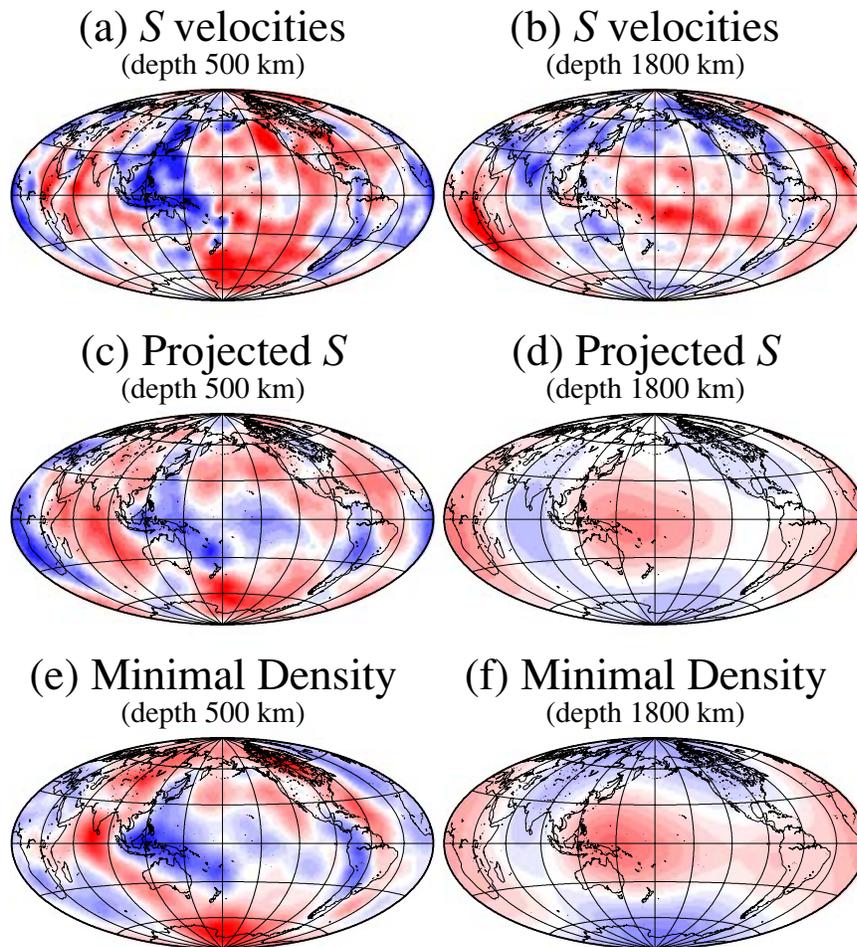
Bearing in mind that the seismic global heterogeneities seen by tomography are more or less proportional to the density anomalies, we project these velocity anomalies onto the  $D_{0lm}$  subset to emphasize the heterogeneities that affect the gravity field of the Earth. To perform this exercise, we use the composite model 'Smean' of Becker & Boschi (2002) (an average of previous models) that has the advantage of providing at the same time a very good fit to the geoid ( $\sim 60$  per cent of variance reduction) and is rich in short-wavelength anomalies (up to  $l = 31$ ). Alternatively, from the observed geoid, we can easily compute the solution of minimal norm for the density. Of course, it does not describe the true density within the Earth but its projection onto the same subset  $D_{0lm}$  as the velocity. The results are depicted in Fig. 3.

The first row depicts the velocity anomalies at mid-upper mantle (a, left) and mid-lower mantle (b, right). The second row filters out from the components invisible to the external gravity field. This filtering emphasizes the anomalies related to slabs in the upper mantle (c). In the lower mantle (d), only the long-wavelength density components affect the external gravity field. These heterogeneities are very close to those predicted from the observed external gravity field (minimal norm density models, bottom row). To have roughly similar colour ranges for the velocity maps (top and middle rows) and for the minimal norm density maps (bottom row), the velocity scale varies between  $\pm 40 \text{ m s}^{-1}$  and the density scale between  $\pm 12 \text{ kg m}^{-3}$ . This suggests throughout the mantle a uniform density/velocity conversion of order  $0.3 \text{ kg m}^3 / \text{m s}^{-1}$ . This conversion factor is typically what is predicted from laboratory estimates and other geoid modelling (e.g. Hager & Clayton 1989).

### 4 CONCLUSIONS

The relationships between the gravity field and the internal structure of the Earth that we have illustrated in this paper are not new. The forward relations have been known since the 1980s (the deep mantle origin of the long-wavelength geoid or degree 2–3, the signature of slabs at degrees 4–10) and various papers have performed inverse modelling. More precisely, it is well known that an appropriate velocity/density conversion of the tomographic models provides a good fit of the geoid. Here we show here that, at each depth, the projections of the velocity  $v_s$  and the density  $\rho$  onto the orthogonal of the gravity null space are proportional. The comparison of the bottom rows of Fig. 3 shows that lateral variations at 1800 km are mainly long-wavelength ones, which is not the case at 500 km. It gives a convincing indication of the deep origin of the geoid at low degrees.

Moreover, our approach can be a useful pedagogical way to explain the non-uniqueness of gravity inversion and our radial basis could be used in seismic tomographic inversion in order to explicitly



**Figure 3.** (a) and (b) Mid-upper and mid-lower mantle  $S$  tomography (Smean model, (Becker & Boschi 2002)). (c) and (d)  $S$  tomographies filtered out from heterogeneities invisible to the gravity field. (e) and (f) Minimal norm densities predicted from the external gravity field. The same scale is used for the four velocity maps (between  $\pm 40 \text{ m s}^{-1}$ ). The two density maps vary between  $\pm 12 \text{ kg m}^{-3}$ .

identify the velocity space constrained by the geoid. The description of the internal density of the Earth requires the knowledge of the coefficients  $\rho_{klm}$ . A lateral resolution  $\lambda_l$  is attained when all the coefficients up to  $l_{\max} = |m_{\max}| \sim 2\pi a/\lambda_l$  are known. A radial resolution  $\lambda_r$  corresponds to  $k_{\max} \sim a/\lambda_r$ . From a given model deduced from seismic modelling and complete up to  $k_{\max}$  and  $l_{\max}$ , i.e. having  $k_{\max}(l_{\max} + 1)^2$  terms, a subset of dimension  $k_{\max}$  could have been deduced directly from the gravity field (assuming that the relationship between velocities and densities are known, and that the viscosity is only depth-dependent). This shows at the same time the potentiality and the low resolution power of gravity.

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