

# Flattening of the Earth: further from hydrostaticity than previously estimated

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## SUMMARY

The knowledge of the gravitational potential coefficients  $J_2$  and  $J_4$  of a hydrostatic Earth model is necessary to deal with non-hydrostatic properties of our planet. They are indeed fundamental parameters when modelling the 3-D density structure or the rotational behaviour of our planet. The most widely used values computed by Nakiboglu need to be updated for two reasons. First, we have noted a mistake in one of his formulae. Secondly, the value of the inertia ratio  $I/MR^2$  chosen at the time of PREM is not any more the best estimate. Both corrections slightly but significantly reduce the hydrostatic  $J_2$  value: the dynamical flattening of the Earth is even further from hydrostaticity than previously thought. The difference between the polar and equatorial radii appears to be  $113 \pm 1$  m (instead of 98 m) larger than the hydrostatic value. Moreover, uncertainties upon the hydrostatic parameters are estimated.

**Key words:** Gravity anomalies and Earth structure; Earth rotation variations; Geopotential theory.

## 1 INTRODUCTION

The equilibrium shape of a rotating, self-gravitating planet is a classical problem of geodesy that dates back to Newton and was studied by the most famous mathematicians like Clairaut, Maclaurin and D'Alembert among many others. The theory of hydrostatic equilibrium predicts the shape and the gravity at the surface of the Earth, as a function of latitude. All results conclude that the hydrostatic flattening of the Earth, with a polar radius about 21 km smaller than its equatorial radius, is indeed close to the observed value (for modern estimates, see e.g. Nakiboglu 1982; Denis 1989). For most practical applications, the reference shape and gravity of the Earth are not based on this theoretical, hydrostatic model but are directly deduced from satellite observations.

There are however geophysical problems where the hydrostatic reference value is important and where the relative agreement between the observed and hydrostatic flattening is not enough. In the geodynamic community, the geoid is not referred to a best-fitting ellipsoid as done in the geodesy community, but to the shape that the Earth should have if gravity and rotation were in equilibrium. This non-hydrostatic geoid only differs at even degrees and order 0 (practically, only at degrees 2 and 4) from those used by geodesists. This non-hydrostatic geoid being most likely induced by the degree-2 order-0 mantle density heterogeneities, the value of the non-hydrostatic  $J_2$  coefficient and of non-hydrostatic flattening of the Earth is used to constrain the modelling of mantle mass anomaly

(e.g. Ricard *et al.* 1984, 1993; Richards & Hager 1984, see also Forte 2007 for a review). The rotational behaviour of our planet after pleistocenian deglaciations is also affected (Mitrovica *et al.* 2005; Cambiotti *et al.* 2010).

For these geophysical questions, a precise estimate of the theoretical hydrostatic geoid is needed, and what most authors have done is to use the theoretical hydrostatic geoid computed by Nakiboglu (1982). It is necessary to reassess this estimate for several reasons. First, since Nakiboglu (1982), the mass and inertia of the Earth have been estimated with higher precisions (Chambat & Valette 2001). As the hydrostatic flattening is controlled by these two quantities and by the radial density profile of the Earth, this impacts the prediction of the hydrostatic shape directly, but also indirectly, by requiring a change of the radial density models of the Earth. For example, PREM model was built in agreement with mass and inertia values that are not those estimated for the Earth any more. Second, the previous attempts do not provide modelling error bars. Thirdly, we discovered a few minor mistakes in previous computations, which affects the numerical estimates of the flattening by quantities larger than the final uncertainty.

## 2 EQUILIBRIUM EQUATIONS

Although the equilibrium equations are given elsewhere, we find it necessary to write them again in this paper and discuss some

differences with Kopal (1960), Lanzano (1982) or Nakiboglu (1982). Like these three authors, we describe the shape of the Earth in terms of flattening. We have not verified the equations of Moritz (1990) given with ellipticity instead of flattening as variable.

The hydrostatic self-gravitational equilibrium theory consists of solving together Poisson's equation  $\Delta\varphi = 4\pi G\rho - 2\Omega^2$  and the hydrostatic equilibrium equation  $\text{grad } p = -\rho \text{ grad } \varphi$  with the boundary conditions  $[\varphi] = 0$ ,  $[\text{grad}\varphi \cdot n] = 0$ ,  $[p] = 0$ ,  $\varphi(x) \sim -\frac{1}{2}(\Omega^2 x^2 - (\Omega \cdot x)^2)$  at  $\infty$ , where  $\varphi$  is here the gravity potential (Newtonian + centrifugal),  $G$  the gravitational constant,  $\rho$  the density,  $p$  the pressure,  $\Omega$  the rotation vector,  $x$  the position vector,  $n$  the unitary normal vector to an interface and where  $[f]$  denotes the jump of a quantity  $f$  across an interface. The hydrostatic equation imposes that equipotential surfaces are also equidensity surfaces. Poisson's equation can be recast into a relation involving one unknown function only, for example, the shape of these surfaces. This relation can then be solved when linearized with respect to a spherical reference.

Explicitly, and correct to second-order, the equation of an equipotential surface  $s = s(r, \theta)$  and the expression of the external gravitational potential  $\phi(r, \theta)$  ( $= -\varphi + \text{centrifugal}$ ) are

$$s(r, \theta) = r(1 + f_2(r)P_2(\cos\theta) + f_4(r)P_4(\cos\theta)), \quad (1)$$

$$\phi(r, \theta) = \frac{GM}{r} \left( 1 - J_2 \frac{a^2}{r^2} P_2(\cos\theta) - J_4 \frac{a^4}{r^4} P_4(\cos\theta) \right), \quad (2)$$

where  $f_n(r)$  and  $J_n$  are non-dimensional factors to be determined. In these equations,  $s$  is the distance from the Earth's centre,  $\theta$  the colatitude ( $s$  and  $\phi$  do not depend on the longitude),  $r$  the mean radius of  $s$  and  $M$  the mass of the Earth. The length  $a$  in (2) is conventional and is usually chosen as the major semi-axis of the reference ellipsoid. We take  $a = 6378137$  m. The  $P_n$  are Legendre polynomials of degree  $n$ , that is,

$$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1), \quad (3)$$

$$P_4(\cos\theta) = \frac{1}{8}(35\cos^4\theta - 30\cos^2\theta + 3). \quad (4)$$

The shape and potential parameters,  $f_n$  and  $J_n$ , are not independent. Taking into account that the external surface (at mean radius  $r = R$ ) is a gravity equipotential, we get

$$J_2 = -\left( f_2(R) + \frac{m}{3} + \frac{11}{7}f_2^2(R) + \frac{m}{7}f_2(R) \right) \left( \frac{a_h}{a} \right)^2, \quad (5)$$

$$J_4 = -\left( f_4(R) + \frac{36}{35}f_2^2(R) + \frac{6m}{7}f_2(R) \right) \left( \frac{a_h}{a} \right)^4, \quad (6)$$

where  $a_h$  is the equatorial semi-axis of the hydrostatic surface,

$$a_h = R \left( 1 - \frac{1}{2}f_2(R) + \frac{3}{8}f_4(R) \right) \quad (7)$$

and where the ratio of centrifugal to gravitational force at mean radius is

$$m = \frac{\Omega^2 R^3}{GM}. \quad (8)$$

The function  $f_n$  can be estimated at any order with respect to the small number  $m$  by integration of differential equations where the variable is  $r$ . The theory was established by Clairaut (1743) at first-order, improved by Airy (1826) and continued by Callandreau (1889) up to second-order and by Lanzano (1962, 1982) up to third-order. For the Earth the second-order is necessary and sufficient.

The primary parameters that enter the computation of  $f_n$  are the angular velocity of the Earth  $\Omega$ , the geocentric gravitational constant  $GM$  and the density distribution of a spherical reference Earth model  $\rho(r)$ . Actually the solution depends on  $m$ , on the mean density  $\bar{\rho}(r)$  within the sphere or radius  $r$  defined by

$$\bar{\rho}(r) = \frac{3}{r^3} \int_0^r \rho(y)y^2 dy, \quad (9)$$

and on the following density factor  $\gamma(r)$

$$\gamma(r) = \frac{\rho(r)}{\bar{\rho}(r)}. \quad (10)$$

The functions  $f_n(r)$  are solutions of the differential system (ch. 2.02 Lanzano 1982):

$$\begin{aligned} r^2 \ddot{f}_2 + 6\gamma r \dot{f}_2 + 6(\gamma - 1)f_2 \\ = \frac{2}{7}(18(1 - \gamma)f_2 + (2 - 9\gamma)r \dot{f}_2)r \dot{f}_2 \\ + 4m \frac{\bar{\rho}(R)}{\bar{\rho}(r)} (1 - \gamma)(f_2 + r \dot{f}_2), \end{aligned} \quad (11)$$

$$\begin{aligned} r^2 \ddot{f}_4 + 6\gamma r \dot{f}_4 + (6\gamma - 20)f_4 \\ = \frac{18}{35} (-21\gamma f_2^2 + 2(2 - 9\gamma)r \dot{f}_2 \dot{f}_2 + (2 - 9\gamma)r^2 \dot{f}_2^2), \end{aligned} \quad (12)$$

where a dot denotes the radial derivative.

We have verified these equations by means of a shape perturbation method as in Chambat & Valette (2001), for the first-order and as in Valette (1987, chap. 5.2) for the second-order. They agree with those given by Nakiboglu (1982). There is a misprint in Kopal's book (1960) in which the coefficient 2 underlined in (12) is replaced by 1.

This differential system must be supplemented by continuity conditions at interfaces and boundary conditions at the centre and at the external surface. The conditions at interfaces are obtained by writing the continuity of the gravity potential and the gravity acceleration, accounting for the non-spherical shape of the interfaces (eq. 1). It results in the continuity of  $f_n$  and  $\dot{f}_n$  across interfaces:

$$[f_2] = [\dot{f}_2] = [f_4] = [\dot{f}_4] = 0. \quad (13)$$

The conditions at the external surface are obtained by writing again the continuity of the gravity potential and acceleration and the fact that the external potential is harmonic (eq. 2). These conditions are

$$\begin{aligned} 2f_2 + R\dot{f}_2 + \frac{5}{3}m \\ = \frac{1}{7} (12f_2^2 + \underline{6}Rf_2\dot{f}_2 + 2R^2\dot{f}_2^2) + \frac{2}{3}m(5f_2 + R\dot{f}_2), \end{aligned} \quad (14)$$

$$4f_4 + R\dot{f}_4 = \frac{18}{35} (6f_2^2 + 5Rf_2\dot{f}_2 + R^2\dot{f}_2^2). \quad (15)$$

These conditions correspond to those written by Kopal (1960) and Lanzano (1982). The underlined factor 6 in (14) is missing in Nakiboglu's article (1982). This is not a misprint since the same mistake appears in Nakiboglu (1979) and since we can reproduce Nakiboglu's numerical results when we use his equation without the factor 6. It matters as we get significantly different results when using (14).

Conditions at the centre arise from the fact that physical fields are regular at this point which is singular in spherical coordinates. For example, the density takes the form  $\rho(r) \simeq \rho(0) + \text{cst } r^2$  in the

vicinity of the centre which implies that  $\gamma \simeq 1 + cst r^2$ . Conditions on  $f_n$  follow from this remark and the hypothesis that  $f_n$  and  $\dot{f}_n$  remain finite. Dividing eq. (11) by  $r$  and making  $r \rightarrow 0$ , every term but  $\dot{f}_2$  vanishes, which leads to

$$\dot{f}_2 = 0. \quad (16)$$

In the same manner, making  $r \rightarrow 0$  in eq. (12), every term but two vanishes, which implies

$$f_4 = \frac{27}{35} f_2^2. \quad (17)$$

Instead of conditions (16) and (17), Nakiboglu and Lanzano write  $f_2 = f_4 = 0$ , which is incorrect. Despite recommendations of Kopal (1960) and Moritz (1990), the integration of the differential system (11–17) is usually performed by using iterative methods. It is in fact simpler to recast the system into a set of two linear systems: one for the first-order and another for the second-order. Indeed a first integration without the terms on the right side of (11)–(15) gives the first-order solution; a second integration, with products of the first-order terms on the right-side of these equations, gives the second-order solution.

To perform the integration easily, a first step is to transform the equations into first-order linear differential systems. This is done in Appendix A. At first-order the system is homogeneous and the numerical integration is straightforward: we integrate the system from the centre where the only physical fundamental solution is  $(f_2, r \dot{f}_2) \simeq (cst, 0)$  up to the surface where the normalization condition (14) provides the constant. At second-order the system is heterogeneous and the resolution proceeds with two integrations: one for a particular solution of the heterogeneous system and one for the general physical fundamental solution of the homogenous system. The normalization of the fundamental solution is obtained by applying the surface condition to the total solution. The integrations were carried out by using a density model in a discretized form, the Runge-Kutta matlab routine ode45 and the matlab spline interpolator interp1 to refine the sampling.

### 3 NUMERICAL RESULTS

#### 3.1 Global Earth data and mean density model

Some global Earth data values are summarized in Table 1. They are taken from Chambat & Valette (2001) who made a thorough analysis of them. Since this publication the only change is the value of  $G$  and consequently of  $M$ , the precision of which has gained a factor 15 (Mohr *et al.* 2008). The  $GM$  value is given here without atmosphere after correction of atmospheric mass  $M_{\text{atm}} = (5.1480 \pm 0.0003) \times 10^{18}$  kg (Trenberth & Smith 2005). This correction significantly affects  $GM$  but does not affect its uncertainty.

Prior density models are not suitable to obtain the best up-to-date estimates of hydrostatic parameters because they do not fit  $R$ ,  $GM$  and  $I/M$  within their error bars. For instance, PREM uses  $R = 6371\,000$  m,  $GM = 3.986\,638\,727 \times 10^{14}$  m<sup>3</sup> s<sup>-2</sup>,  $I/MR^2 = 0.330\,800$ , which implies  $m = 3.449\,236 \times 10^{-3}$  (compare with the actual data and uncertainties in Table 1). As a radial density model, we therefore use a new unpublished mean density model that adjusts, within the observational uncertainties, the Earth radius  $R$ , mass  $M$ , the inertia ratio  $I/M$  (Table 1) and the seismic modes mean frequencies (Valette & Lesage, personal communication). This model remains close to PREM, however, and we will see in Section 3.3 how the bias of PREM can be accounted for.

After integration of the differential system (11)–(12) with this density model we find, correct to first-order (Table 1),

$$J_2^1 = 1.072\,3 \times 10^{-3}, \quad (18)$$

and correct to second-order

$$J_2 = 1.071\,2 \times 10^{-3}, \quad (19)$$

$$J_2 - J_2^1 = 1.085 \times 10^{-6}, \quad (20)$$

$$J_4 = -2.96 \times 10^{-6}. \quad (21)$$

**Table 1.** Data for reference Earth model. The values in parenthesis are the uncertainties referred to the last figures of the nominal values.

Data	Symbol	Value (uncertainty)	Unit	Relative uncertainty
Observed <sup>a</sup>				
Physical mean radius	$R$	6.371 230 (10)	10 <sup>6</sup> m	$1.6 \times 10^{-6}$
Geocentric gravitational constant <sup>b</sup>	$GM$	3.986 000 979 (40)	10 <sup>14</sup> m <sup>3</sup> s <sup>-2</sup>	$1.0 \times 10^{-8}$
Angular velocity	$\Omega$	7.292 115 0 (1)	10 <sup>-5</sup> rad s <sup>-1</sup>	$1.4 \times 10^{-8}$
Rotationnal factor	$m$	3.450 162 (16)	10 <sup>-3</sup>	$4.7 \times 10^{-6}$
Gravitational constant	$G$	6.674 28 (67)	10 <sup>-11</sup> m <sup>3</sup> kg <sup>-1</sup> s <sup>-2</sup>	$1.0 \times 10^{-4}$
Mass	$M$	5.972 18 (60)	10 <sup>24</sup> kg	$1.0 \times 10^{-4}$
Inertia ratio <sup>b,c</sup>	$I/M$	1.342 354 (31)	10 <sup>13</sup> m <sup>2</sup>	$2.3 \times 10^{-5}$
Inertia coefficient <sup>b,c</sup>	$I/MR^2$	0.330 690 (9)		$2.6 \times 10^{-5}$
Degree 2 zonal potential coefficient <sup>b,d,e</sup>	$J_2 _{\text{obs-corr}}$	1.082 604 6 (5)	10 <sup>-3</sup>	$4.6 \times 10^{-7}$
Degree 4 zonal potential coefficient <sup>e</sup>	$J_4$	-1.620 (1)	10 <sup>-6</sup>	$6.2 \times 10^{-4}$
Hydrostatic (this study)				
Fluid degree two Love number	$k$	0.932 33 (9)		$1 \times 10^{-4}$
Degree 2 zonal potential coefficient, first-order <sup>e</sup>	$J_2^1$	1.072 3 (1)	10 <sup>-3</sup>	$1 \times 10^{-4}$
Degree 2 zonal potential coefficient, second-order <sup>e</sup>	$J_2$	1.071 2 (1)	10 <sup>-3</sup>	$1 \times 10^{-4}$
Difference of second- and first-order <sup>e</sup>	$J_2 - J_2^1$	-1.085 (3)	10 <sup>-6</sup>	$3 \times 10^{-3}$
Degree 4 zonal potential coefficient <sup>e</sup>	$J_4$	-2.96 (3)	10 <sup>-6</sup>	$1 \times 10^{-2}$

<sup>a</sup>From Chambat & Valette (2001) with modifications explained in text.

<sup>b</sup>Without atmosphere.

<sup>c</sup>Inertia ratio of the spherical model that is closest to the Earth.

<sup>d</sup>Without direct and hydrostatic indirect permanent tide.

<sup>e</sup> $J_2$  and  $J_4$  are scaled with  $GM$  given in this table and  $a = 6378\,137$  m.

### 3.2 Uncertainties

The uncertainties in the computed hydrostatic values are given in Table 1 and have been evaluated in the following way. At first-order we define the so-called degree 2 fluid Love number  $k$  by

$$f_2(R) = -(k+1) \frac{m}{3}. \quad (22)$$

Then relations (5) and (14) can be written as

$$J_2^1 = k \frac{m}{3} \left( \frac{a_h}{a} \right)^2 \quad (23)$$

and

$$k = \frac{3 - R \dot{f}_2 / f_2}{2 + R \dot{f}_2 / f_2} (R). \quad (24)$$

Relation (22) implies that the uncertainty of the first-order hydrostatic theory is essentially controlled by the one of  $k$  since  $m$  is much better known (see Table 1).

A result from Radau (1885) shows that  $k$  depends upon the density essentially through  $I/MR^2$ . Indeed to a very good approximation we have (see e.g. Dahlen & Tromp 1998, p. 599-600)

$$k \simeq k_{\text{Radau}} = \frac{5}{\frac{25}{4} \left( 1 - \frac{3}{2} \frac{I}{MR^2} \right)^2 + 1} - 1. \quad (25)$$

From the error of  $I/MR^2$  given in Table 1 and Radau's formula (25) a relative error of  $7 \times 10^{-5}$  is found for  $k$ . To improve this estimation, we also compute  $k$  with various density models, obtained by perturbing our reference density profile while keeping  $I/MR^2$  constant. We achieve that by varying the density and the interfaces radii. These tests show that changing the internal density at constant  $I/MR^2$  affects  $k$  by certainly less than  $3 \times 10^{-5}$  in relative value. A conservative value of the relative uncertainty on  $k$  and  $J_2^1$  is therefore  $10^{-4}$ .

The uncertainties of  $J_2 - J_2^1$  and  $J_4$  are also estimated with the dispersions obtained by using different density models. As can be seen in Table 1 the uncertainty of  $J_2$  is practically equal to that of  $J_2^1$ .

### 3.3 Validation and corrections

It was not possible to compare exactly our results with Nakiboglu's article (1982) because he did not give the value of  $m$  he used. We have compared our results with those of Denis (1989), using his value of  $m = 0.00345039$  and PREM model. The integration gives

$$f_2 = -2.228947 \times 10^{-3}, \quad f_4 = 4.445 \times 10^{-6}, \quad (26)$$

$$k = 0.93311, \quad J_2 = 1.0721 \times 10^{-3}, \quad (27)$$

while Denis found

$$f_2 = -2.228946 \times 10^{-3}, \quad f_4 = 4.465 \times 10^{-6}, \quad (28)$$

which agrees with our values taking the uncertainties into account. Denis did not give estimations for  $k$  or  $J_2$ .

Nakiboglu's (1982) results,  $J_2 = 1.0727 \times 10^{-3}$  and  $J_4 = -2.99 \times 10^{-6}$ , differ respectively from our values by  $15 \times 10^{-7}$  and  $0.3 \times 10^{-7}$ , which are 15 and 1 times our uncertainties. His mistake in the  $J_2$  estimate, due to the missing factor 6 in eq. (14), accounts for  $8 \times 10^{-7}$  and the difference in  $I/MR^2$  for  $9 \times 10^{-7}$ . The remaining discrepancy of  $-2 \times 10^{-7}$  should be explained by a difference in  $m$ .

Note that  $I/MR^2$  is the parameter that influences  $k$  the most and that this influence can be quantified by means of Radau's theory.

Thus, we can correct the fact that the used density model does not correspond to the observed  $I/MR^2$  through

$$k_{\text{corrected}} = k + k_{\text{Radau}} \left( \frac{I/MR^2|_{\text{observed}}}{I/MR^2|_{\text{model}}} \right) - k_{\text{Radau}} \left( \frac{I/MR^2|_{\text{model}}}{I/MR^2|_{\text{observed}}} \right). \quad (29)$$

For instance, applying this correction to the  $k$  of PREM (27) yields

$$k_{\text{corrected}} = 0.93232 \quad (30)$$

and with the up-to-date value for  $m$  (Table 1)

$$J_2^{\text{corrected}} = 1.0712 \times 10^{-3} \quad (31)$$

which, taking the uncertainties into account, correspond to our values.

### 3.4 Comparison to observations

For the sake of comparison with a hydrostatic value, the most suitable  $J_2|_{\text{obs-corr}}$  (see Table 1) is the observed value  $J_2|_{\text{obs}}$ , excluding both the atmosphere contribution  $J_2|_{\text{atm}}$  and the permanent direct,  $\Delta J_2$ , and indirect,  $k\Delta J_2$ , luni-solar tide effects:

$$J_2|_{\text{obs-corr}} = J_2|_{\text{obs}} - \Delta J_2(1+k) - J_2|_{\text{atm}}. \quad (32)$$

For a permanent tide, the degree 2 fluid Love number  $k = 0.93233$  is appropriate even if most geodetic publications seem to use an elastic Love number of 0.3. We take  $J_2|_{\text{obs}} = 1.0826264 \times 10^{-3}$  and  $\Delta J_2 = 3.1108 \times 10^{-8}$  as in Chambat & Valette (2001). To compute the atmospheric contribution  $J_2|_{\text{atm}}$  we consider the atmosphere as an homogeneous infinitely thin layer as done in Appendix B. Finally, the permanent tide and atmospheric corrections represent respectively 60 and 4 times the observational uncertainty.

The hydrostatic  $J_2$  predicted in this paper and Nakiboglu's one differ from the observed one by  $114 \times 10^{-7}$  and  $99 \times 10^{-7}$ , respectively. Our new hydrostatic  $J_2$  value is further away from the observed one than that of Nakiboglu by about 15 per cent. The Earth is more flattened than the hydrostatic model. With the above values we find that the difference between the equatorial and polar semi-axis of the Earth exceeds by about  $113 \pm 1$  m the hydrostatic prediction while Nakiboglu's estimation was 98 m.

In studies of postglacial true polar wander, one currently uses the difference between the 'observed'  $k$  (deduced from 23 using the observed  $J_2$ ) and the hydrostatic  $k$ . For that coefficient, denoted  $\beta$  by Mitrovica *et al.* (2005), we recommend a value of  $0.0097 \pm 0.0001$ .

### 3.5 Conclusion

We have updated the values and uncertainties of hydrostatic Love number  $k$  and gravitational potential coefficients  $J_2$  and  $J_4$ . The new

**Table 2.** Normalised potential coefficients  $\bar{C}_{\ell m} = -J_{\ell} / \sqrt{2\ell + 1}$ . They are scaled with  $GM$  given in Table 1 and  $a = 6378137$  m. The values in parentheses are the uncertainties referred to the last figures of the nominal values.

	$\bar{C}_{20} \times 10^6$	$\bar{C}_{40} \times 10^6$
Observed	-484.155 5 (2)	0.540 0 (3)
Hydrostatic this study	-479.06 (5)	0.986 (10)
Hydrostatic Nakiboglu (1982)	-479.73	0.997
Difference observed - hydrostatic from this study	-5.10 (5)	-0.446 (10)
Difference observed - hydrostatic Nakiboglu (1982)	-4.43	-0.457

$J_2$  value is further from the observed one than the currently used of Nakiboglu by about 15 per cent. For the non-hydrostatic geoid we recommend to use the values of normalized potential coefficients  $\bar{C}_{20} = (-5.10 \pm 0.05) \times 10^{-6}$  and  $\bar{C}_{40} = (-0.446 \pm 0.010) \times 10^{-6}$  (see Table 2).

The authors' MATLAB package that solves Clairaut's equations is available at <http://frederic.chambat.free.fr/hydrostatic>.

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## APPENDIX A: FIRST-ORDER SYSTEMS

Correct to second-order,  $f_2$  can be written as a sum of a first- and a second-order term. At first-order we define

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} f_2 \\ r \dot{f}_2 \end{pmatrix} \quad (\text{A1})$$

and at second-order

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} f_2 - y_1 \\ r \dot{f}_2 - y_2 \end{pmatrix} \quad (\text{A2})$$

$$\tilde{z} = \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} f_4 \\ r \dot{f}_4 \end{pmatrix}. \quad (\text{A3})$$

Then, using eqs (11–13) it is easy to show that  $y$ ,  $z$  and  $\tilde{z}$  are continuous at the interfaces:  $[y] = [z] = [\tilde{z}] = 0$  and are solutions of the following differential systems:

– at first-order

$$\dot{y} = Ay, \quad (\text{A4})$$

where

$$A = \frac{1}{r} \begin{pmatrix} 0 & 1 \\ 6(1 - \gamma) & 1 - 6\gamma \end{pmatrix}, \quad (\text{A5})$$

with, at the centre,

$$y_2 = 0 \quad (\text{A6})$$

and, at the external surface,

$$2y_1 + y_2 = -\frac{5}{3}m. \quad (\text{A7})$$

– at second-order

$$\dot{z} = Az + B, \quad \dot{\tilde{z}} = \tilde{A}\tilde{z} + \tilde{B}, \quad (\text{A8})$$

where

$$\tilde{A} = \frac{1}{r} \begin{pmatrix} 0 & 1 \\ 20 - 6\gamma & 1 - 6\gamma \end{pmatrix}, \quad (\text{A9})$$

$$B = \begin{pmatrix} 0 \\ B_2 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 0 \\ \tilde{B}_2 \end{pmatrix}, \quad (\text{A10})$$

$$B_2 = \frac{2y_2}{7r} (9(2 - \gamma)y_1 + (2 - 9\gamma)y_2) + 4m \frac{\bar{\rho}(R)}{\bar{\rho}(r)} (1 - \gamma)(y_1 + y_2), \quad (\text{A11})$$

$$\tilde{B}_2 = \frac{18}{35r} (2y_2(2y_1 + y_2) - 3\gamma (7y_1^2 + 6y_1y_2 + 3y_2^2)), \quad (\text{A12})$$

with, at the centre,

$$z_2 = 0, \quad (\text{A13})$$

$$z_3 = \frac{27}{35}y_1^2, \quad (\text{A14})$$

and, at the external surface,

$$2z_1 + z_2 = \frac{2}{7} (6y_1^2 + 3y_1y_2 + y_2^2) + \frac{2}{3}m (5y_1 + y_2), \quad (\text{A15})$$

$$4z_3 + z_4 = \frac{18}{35} (6y_1^2 + 5y_1y_2 + y_2^2). \quad (\text{A16})$$

In both cases the potential coefficients are given by formula (5–7), retaining only the terms of appropriate order.

## APPENDIX B: ATMOSPHERIC CORRECTION FOR $J_2$

The coefficient  $J_2$  is linked to the Earth's density by (e.g. Chambat & Valette 2001)

$$-Ma^2 J_2 = \int_{\text{Earth}} \rho r^4 P_2 \sin \theta \, dr \, d\theta \, d\lambda. \quad (\text{B1})$$

Suppose that the atmosphere is homogeneous with density  $\rho$  and bounded by the surfaces  $s_-(r_-, \theta, \lambda)$  and  $s_+(r_+, \theta, \lambda)$ . The atmospheric contribution in  $J_2$  is then given by

$$-Ma^2 J_{2|\text{atm}} = \frac{1}{5} \rho \int_{S_1} (s_+^5 - s_-^5) P_2 \, d\omega, \quad (\text{B2})$$

where  $S_1$  denotes the sphere of unit radius and  $\omega$  the solid angle. Suppose that the mean atmospheric thickness  $\Delta R$  is small, then

$$-Ma^2 J_{2|\text{atm}} = \frac{1}{5} \rho \Delta R \int_{S_1} \left. \frac{ds^5}{dr} \right|_{r=R} P_2 \, d\omega. \quad (\text{B3})$$

Correct to first-order,  $s(r, \theta) = r\{1 + f_2(r)P_2(\cos \theta)\}$ , and thus

$$-Ma^2 J_{2|\text{atm}} = \rho \Delta R \int_{S_1} R^4 \{1 + 5f_2 P_2 + R \dot{f}_2 P_2\} P_2 \, d\omega. \quad (\text{B4})$$

Now, by using the properties of Legendre polynomials and the definition of the atmospheric mass

$$\int_{S_1} P_2 \, d\omega = 0, \quad \int_{S_1} P_2^2 \, d\omega = \frac{4\pi}{5}, \quad M_{\text{atm}} = 4\pi \rho R^2 \Delta R, \quad (\text{B5})$$

we deduce that

$$-Ma^2 J_{2|\text{atm}} = M_{\text{atm}} R^2 (f_2 + R \dot{f}_2 / 5)(R). \quad (\text{B6})$$

In order to estimate this value and because of its smallness, we can suppose that the atmosphere and the solid Earth are in hydrostatic equilibrium. Then relations (22–24) yield

$$\begin{aligned} J_{2|\text{atm}} &= \frac{M_{\text{atm}} R^2}{M} \frac{8 + 3k}{a_h^2} \frac{J_2^1}{5k} \simeq \frac{M_{\text{atm}}}{M} \frac{8 + 3k}{5k} J_2^1 \\ &\simeq 2.0 \times 10^{-6} J_2^1 \simeq 2.1 \times 10^{-9}. \end{aligned} \quad (\text{B7})$$

The  $J_{2|\text{atm}}$  value must be subtracted from the observed  $J_{2|\text{obs}}$  in order to remove the atmospheric effect.